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ЭЛЕМЕНТЫ СУПЕРСИММЕТРИЧНОЙ ТЕОРИИ ПОЛЯ

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Пособие представляет собой краткий курс лекций по N=1, D=4 глобально суперсимметричной теории поля, сформулированной в терминах суперполей. Рассматриваются элементы теории групп Лоренца и Пуанкаре, вводятся двухкомпонентные спиноры, суперпространство и суперполя, обсуждаются модель Весса-Зумино, суперсимметричная сигма-модель, суперсимметричная теория поля Янга-Миллса, суперполевая теория возмущений и теорема о неперенормировке. Материал рассчитан на читателя, который ранее не изучал суперсимметрию, но хотел бы познакомиться с с основными понятиями и методами суперсимметричной теории поля.

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ELEMENTS OF SUPERSYMMETRIC FIELD THEORY

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Preface

These notes are extended version of the lectures which I have given at International Advanced School on Modern Mathematical Physics, JINR, Dubna, July 14-16, 2005; at Summer School "Physics of Fundamental Interactions", Protvino, August 17-26, 2006; at IV International Summer School in Modern Mathematical Physics, Belgrade, September 3-14, 2006; and at Brazilian Center for Fundamental Physics (CBPF), Rio de Janeiro, January, 2006. They devoted to brief introduction to the basic notions of $\mathcal{N} = 1$ $D = 4$ supersymmetry, construction of supersymmetric field models and some of their quantum aspects. From the very beginning the material is presented completely in manifestly supersymmetric form in terms of superspace and superfields.

These notes can also be entitled as "Pedagogical Introduction to Supersymmetry" since the material has an educational character and is given in the form allowing the interested reader to acquire independently the first notations of supersymmetric filed theory. No any preliminary knowledge on supersymmetry is assumed though it is expected that the readers passed the standard course of quantum field theory.

We consider here only basic and simple enough notions of supersymmetric filed theory. The more advanced aspects are left for further study. The theoretical material is supplemented with a number of problems which should help the readers to practice a formalism of supersymmetric field theory.

I am grateful to Organizers of all above meetings, especially to Professor B. Dragovich (Institute of Physics, Belgrade), Professor A. Filippov (Joint Institute for Nuclear Research, Dubna), Professor J. Helayel-Neto (Brazilian Center for Fundamental Physics, Rio de Janeiro), Professor M. Vasiliev (Lebedev Physical Institute, Moscow) and Dmitry Zimin Foundation "Dynasty" (Moscow) for support and possibility to deliver the lecture courses and kind atmosphere during the meetings. Also, I am grateful to all participants of the meetings for the interest to my lectures. The invaluable help in preparation of computer file has been rendered by Denis Malishev and Igor Samsonov.

1 General idea of supersymmetry

Supersymmetry in physics means a hypothetical symmetry of Nature relating the bosons and fermions. In its essence, the supersymmetry is an extension of special relativity symmetry. One can say, the supersymmetry is a special relativity symmetry extended by the symmetry between bosons and fermions.

The main idea of supersymmetry in field theory can be explained as follows.

Let us consider some model of field theory. Any such model is given in terms of an action functional, $S[b, f]$, depending on a set of bosonic fields $b(x)$ and a set of fermionic fields $f(x)$. Consider the infinitesimal transformations of the fields $b(x)$ and $f(x)$ of the form

$$\begin{aligned} b &\rightarrow b + \delta b, \qquad && \delta b \sim f, \\ f &\rightarrow f + \delta f, \qquad && \delta f \sim b. \end{aligned} \tag{1.1}$$

If the action $S[b, f]$ is invariant under the transformations (1.1), $\delta S[b, f] = 0$, the field model under consideration is called *supersymmetric*. The transformations (1.1) are called *supersymmetry transformations* or *supertransformations*.

Of course, the above statements look like very schematic and naive while we have not answered the following questions:

1. What are an explicit sets of the fields b and f in the model under consideration?

2. What is an explicit form of transformations (1.1)?

3. What is an explicit form of the invariant action $S[b, f]$?

Originally, the supersymmetry was proposed in 1971 by Yu. Golfand and E. Lichtman from Lebedev Physical Institute. It was then rediscovered again in some another form in 1972 by D. Volkov and V. Akulov from Khar'kov Institute of Physics and Technology. In 1974 J. Wess and B. Zumino from Karlsruhe University and CERN respectively, constructed the first supersymmetric model of four-dimensional field theory. After that, a number of papers on supersymmetry became to increase as a rolling snow ball. It is also worth pointing out a proposal of two-dimensional supersymmetry in 1971 in the context of string theory (P. Ramond, A. Neveu, J. Schwarz, J. Gervais, B. Sakira). Further, we will discuss only four-dimensional supersymmetric field theories.

If the supersymmetry is a true symmetry of Nature, it immediately leads to the fundamental physical consequences. According to the Standard Model, all elementary particles form two classes: the particles of matter which are the fermions and the particles mediating the fundamental interactions which are the bosons. However, if the concept of supersymmetry is true, the classification of fundamental particles on bosons and fermions is relative since the supersymmetry transforms bosons into fermions and vice versa. It means that for each boson there should exist a corresponding superpartner – a fermion and for each fermion there should exist a corresponding superpartner – a boson. Hence, for each fermionic matter particle there must exist a bosonic matter particle and for each bosonic particle – mediator of fundamental interaction there must exist a fermionic particle – mediator of fundamental interaction. As a result, one gets a beautiful symmetric picture of Nature on the fundamental level.

As we already mentioned, up to now the supersymmetry is a hypothetical symmetry since there is no experimental evidence for it at present. However, the experiments for searching the superpartners of known particles have already been planned for the next several years. Further we are not going to discuss the experimental aspects of supersymmetry.

As is already pointed out, the supersymmetry is an extension of special relativity symmetry. Special relativity is based on invariance of physical phenomena under the Lorentz transformations and four-dimensional translations $x'^m = x^m + a^m$, where x^m are the Minkowski space coordinates and a^m is a constant four-vector, $m = 0, 1, 2, 3$. Supersymmetry includes special relativity symmetry and some additional symmetry associated with the transformations of the form (1.1) and responsible for relations between bosons and fermions. There is a universal procedure to realize the special relativity transformations together with the transformations (1.1). Such a procedure is based on the notions of superspace and superfield. Our purpose is to explain the meaning of this notions, to show, how the supersymmetric field models are constructed in terms of superfields and to demonstrate the technics of operation with superfields. The notations and conventions correspond to the book I.L. Buchbinder, S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity*, IOP Publ., 1998.

2 Lorentz and Poincare groups

2.1 Basic definitions

Let us consider the four-dimensional Minkowski space with the coordinates x^m, $m = 0, 1, 2, 3$ and the metric

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \equiv \eta_{mn} dx^m dx^n, \quad (2.1)$$

where

$$\eta = \mathrm{diag}(-1, 1, 1, 1). \quad (2.2)$$

It is easy to see, that the form of the metric (2.1) is invariant under the following linear inhomogeneous transformations

$$x^m \longrightarrow x'^m = \Lambda^m{}_n x^n + a^m, \quad (2.3)$$

where a^m is a constant four-vector and $\Lambda = (\Lambda^m{}_n)$ is some matrix with constant elements. To find the restrictions on this matrix one considers form invariance condition

$$\eta_{mn} dx'^m dx'^n = \eta_{pq} dx^p x^q. \quad (2.4)$$

Substituting the transformations (2.3) into Eq. (2.4), one gets

$$\eta_{mn}\Lambda^m{}_p\Lambda^n{}_q dx^p dx^q = \eta_{pq} dx^p dx^q, \tag{2.5}$$

or

$$\eta_{mn}\Lambda^m{}_p\Lambda^n{}_q = \eta_{pq}. \tag{2.6}$$

The equation (2.6) can also be rewritten in the matrix form

$$\Lambda^{\mathrm{T}}\eta\Lambda = \eta, \tag{2.7}$$

where Λ^{T} is a transpose matrix with the elements $(\Lambda^{\mathrm{T}})_n{}^m = \Lambda^m{}_n$. The transformations (2.3) with the matrix Λ satisfying the constraint (2.7) are called the *inhomogeneous Lorentz transformations.*

Clearly, the set of transformations (2.3) is given by a pair (a, Λ) with Λ being the Lorentz matrix satisfying Eq. (2.7) and a constant four-vector a. The set of such transformations (a, Λ) has two natural subsets:

i) The subset of transformations

$$x'^m = x^m + a^m \tag{2.8}$$

is given by the pair $(a, \mathbb{I})$, where $\mathbb{I}$ is the unit 4×4 matrix. The transformations (2.8) are called *space-time translations.*

ii) The subset of transformations

$$x'^m = \Lambda^m{}_n x^n, \tag{2.9}$$

with Λ satisfying Eq. (2.7), is given by the pair $(0, \Lambda)$. The transformations (2.9) are called *Lorentz rotations* or *homogeneous Lorentz transformations.*

Let us consider two inhomogeneous Lorentz transformations of the form (2.3) given by the pairs (a_1, Λ_1) and (a_2, Λ_2). Performing these transformations consequently, one arrives at the relation

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (\Lambda_2 a_1 + a_2, \Lambda_2 \Lambda_1). \tag{2.10}$$

One can show that if the matrices Λ_1 and Λ_2 satisfy Eq. (2.7), their product $\Lambda_2\Lambda_1$ satisfies Eq. (2.7) as well. Therefore, Eq. (2.10) means that the composition of two inhomogeneous Lorentz transformations is also an inhomogeneous Lorentz transformations.

It is easy to see now that the set of transformations (a, Λ) forms a group with the multiplication law given by Eq. (2.10). Indeed, the unit group element is $(0, \mathbb{I})$ and the element inverse to (a, Λ) is $(-\Lambda^{-1}a, \Lambda^{-1})$. Here Λ^{-1} is the inverse matrix satisfying the relation

$$\Lambda^{-1}\Lambda = \Lambda\Lambda^{-1} = \mathbb{I} \tag{2.11}$$

or

$$\Lambda^m{}_p(\Lambda^{-1})^p{}_n = \delta^m{}_n. \tag{2.12}$$

This group is called the *Poincare group*. One can show that the subset of transformations $(0, \Lambda)$ forms the group which is called the *Lorentz group*. The subset of transformations $(a, \mathbb{I})$ also forms a group which is called the *translation group*. Two these groups are the subgroups of the Poincare group.

One can check the following useful identity

$$(a, \Lambda) = (a, \mathbb{I})(0, \Lambda), \tag{2.13}$$

which means that any element of the Poincare group is represented as the composition of two elements from the Lorentz group and the translation group.

Further we will use only the infinitesimal form of inhomogeneous Lorentz transformations. For this purpose we represent the Lorentz matrix as $\Lambda = \mathbb{I} + \omega$, where $\omega \equiv (\omega^m{}_n)$ is the matrix with infinitesimal elements. In this case basic relation (2.7) takes the following form

$$(\mathbb{I} + \omega)^{\mathrm{T}} \eta (\mathbb{I} + \omega) = \eta, \tag{2.14}$$

or

$$\omega_{mn} + \omega_{nm} = 0, \tag{2.15}$$

where $\omega_{mn} = \eta_{mp} \omega^p{}_n$. The equation (2.15) means that ω_{mn} is an arbitrary antisymmetric matrix which has six independent real elements. Therefore the Lorentz group is a six-parametric Lie group while the Poincare group is a ten-parametric Lie group.

2.2 Proper Lorentz group and SL(2,$\mathbb{C}$) group

The basic relation (2.7) for matrices Λ leads to the following identity

$$\det \Lambda^{\mathrm{T}} \det \eta \det \Lambda = \det \eta. \tag{2.16}$$

As a consequence,

$$\det \Lambda = \pm 1. \tag{2.17}$$

In particular, Eq. (2.17) means that any Lorentz matrix is non-degenerate and invertible. Moreover, one can show that the element $\Lambda^0{}_0$ is either positive or negative

$$\mathrm{sign}(\Lambda^0{}_0) = \pm 1. \tag{2.18}$$

Equations (2.17) and (2.18) show that the set of matrices Λ is divided into four non-overlapping subsets. Here we are interested only in one such subset denoted as $L_+^\uparrow$ and defined by the conditions

$$\Lambda \in L_+^\uparrow \quad \Leftrightarrow \quad \Lambda^{\mathrm{T}}\eta\Lambda = \eta,\ \det\Lambda = 1,\ \mathrm{sign}\Lambda^0{}_0 = 1. \tag{2.19}$$

One can show, that the set of transformations $(0, \Lambda)$, where $\Lambda \in L_+^\uparrow$, forms a group which is called *proper Lorentz group.*

Further we will show that the proper Lorentz group allows us to introduce the specific objects which are called the two-component spinors.

First of all, we introduce a set of 2×2 complex matrices N with unit determinant, $\det N = 1$. Clearly, the set of such matrices forms a group with the multiplication law being an ordinary matrix product. This group is called two-dimensional special linear complex group and denoted as $\mathrm{SL}(2,\mathbb{C})$.

One can show that for each matrix $N \in \mathrm{SL}(2,\mathbb{C})$ there exists the matrix $\Lambda \in L_+^\uparrow$ such that

i) $\Lambda(N_1N_2) = \Lambda(N_1)\Lambda(N_2)$,

ii) $\Lambda(N_1) = \Lambda(N_2) \Leftrightarrow N_1 = \pm N_2$.

The construction of the map $\Lambda(N)$ is realized in the following five steps.

1. Consider a linear space of Hermitian 2×2 matrices X, $X^+ = X$. The basis in this space can be chosen in form of following four matrices $\sigma_m \equiv (\sigma_0, \vec{\sigma})$:

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \sigma_2 &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.20}$$

Here the σ_1, σ_2, σ_3 are very well known Pauli matrices and σ_0 is the unit 2×2 matrix. Any Hermitian matrix X can be expanded over the basis (2.20) as

$$X = X^m \sigma_m, \tag{2.21}$$

where X^m are the real numbers. It is easy to check the following identity for the matrices (2.20)

$$\mathrm{tr}(\sigma_m \sigma_n) = 2\delta_{mn}, \tag{2.22}$$

which helps to express the coordinates X^m as

$$X^m = \frac{1}{2}\mathrm{tr}(X\sigma_m). \tag{2.23}$$

2. Let $N \in \mathrm{SL}(2,\mathbb{C})$. Consider the matrix

$$X' = NXN^+. \tag{2.24}$$

Since $\det N = 1$, one gets

$$\det X' = \det X. \tag{2.25}$$

3. Using equations (2.23) and (2.25) one obtains

$$\begin{aligned} X'^m &= \frac{1}{2}\mathrm{tr}(X'\sigma_m) = \frac{1}{2}\mathrm{tr}(NXN^+\sigma_m) \\ &= \frac{1}{2}\mathrm{tr}(\sigma_m N\sigma_n N^+)X^n \equiv \Lambda^m{}_n X^n, \end{aligned} \tag{2.26}$$

where

$$\Lambda^m{}_n = \Lambda^m{}_n(N) = \frac{1}{2}\mathrm{tr}(\sigma_m N\sigma_n N^+). \tag{2.27}$$

4. Using the basis matrices (2.20), one rewrite the expansion $X = X^m\sigma_m$ as follows

$$X = \begin{pmatrix} X^0 + X^3 & X^1 - iX^2 \\ X^1 + iX^2 & X^0 - X^3 \end{pmatrix}. \tag{2.28}$$

The straightforward calculation of the determinant of the matrix X gives

$$\det X = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 = -\eta_{mn}X^m X^n. \tag{2.29}$$

The same consideration leads to

$$\det X' = -\eta_{mn}X'^m X'^n. \tag{2.30}$$

Applying now the equation (2.25), we have

$$\eta_{mn}X^m X^n = \eta_{mn}X'^m X'^n. \tag{2.31}$$

Substituting Eq. (2.26) into (2.31), one gets

$$\Lambda^{\mathrm{T}}\eta\Lambda = \eta, \tag{2.32}$$

where Λ is given by (2.27). This means that the matrix Λ is nothing but a matrix of Lorentz rotation.

5. One can show that Eq. (2.27) leads to

$$\Lambda^0{}_0 > 0, \qquad \det\Lambda = 1. \tag{2.33}$$

Thus, the matrices Λ (2.27) belong to the proper Lorentz group.

As a result, we have proved that the proper Lorentz group $L_+^\uparrow$ is associated with $\mathrm{SL}(2,\mathbb{C})$ group.

2.3 Two-component spinors

It is evident that the complex 2×2 matrices $N \in \mathrm{SL}(2,\mathbb{C})$ act in two-dimensional complex space. We denote the vectors of this space as φ_α, $\alpha = 1,2$. Action of these matrices on the vectors φ_α looks like

$$\varphi'_\alpha = N_\alpha{}^\beta \varphi_\beta. \tag{2.34}$$

Since each matrix $N \in \mathrm{SL}(2,\mathbb{C})$ is associated with some matrix Λ from the proper Lorentz group, one can say that Eq. (2.34) is the transformation law of the two-dimensional complex vector under the Lorentz transformations. The vectors φ_α with the transformation law (2.34) are called *left Weyl spinors*. The indices α, β are called the spinor ones.

Let us introduce the matrix ε with elements $\varepsilon_{\alpha\beta}$,

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.35}$$

and its inverse matrix ε^{-1} with the elements $\varepsilon^{\alpha\beta}$, that is

$$\varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma, \quad \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^\alpha_\gamma, \quad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.36}$$

One can prove the following identities

$$N\varepsilon N^{\mathrm{T}} = \varepsilon, \quad N^{\mathrm{T}}\varepsilon^{-1}N = \varepsilon^{-1} \tag{2.37}$$

for arbitrary matrix $N \in \mathrm{SL}(2,\mathbb{C})$. The equation (2.37) means that the matrices ε and ε^{-1} are the invariant tensors of $\mathrm{SL}(2,\mathbb{C})$ group. Therefore these matrices can be used for rising and lowering the spinor indices

$$\varphi^{\alpha} = \varepsilon^{\alpha\beta}\varphi_{\beta}, \qquad \varphi_{\alpha} = \varepsilon_{\alpha\beta}\varphi^{\beta}. \tag{2.38}$$

Moreover, one can show that the expression $\varphi_1^{\alpha}\varphi_{2\alpha}$ is Lorentz invariant, i.e.,

$$\varphi_1'^{\alpha}\varphi_{2\alpha}' = \varphi_1^{\alpha}\varphi_{2\alpha}, \tag{2.39}$$

where φ_{α}' is given by (2.34) and $\varphi'^{\alpha} = \varepsilon^{\alpha\beta}\varphi_{\beta}'$.

Consider the matrix N^* which is complex conjugate to the matrix N, $N \in \mathrm{SL}(2,\mathbb{C})$. It is clear, the matrix N^* belongs to the group $\mathrm{SL}(2,\mathbb{C})$ as well, $N^* \in \mathrm{SL}(2,\mathbb{C})$. We denote the elements of the matrix N^* as $N^{*\dot{\beta}}_{\dot{\alpha}}$; $\dot{\alpha},\dot{\beta} = \dot{1},\dot{2}$. This matrix acts in the complex space of two-dimensional vectors $\chi_{\dot{\alpha}}$ by the rule

$$\chi_{\dot{\alpha}}' = N^{*\dot{\beta}}_{\dot{\alpha}}\chi_{\dot{\beta}}. \tag{2.40}$$

The two-component complex vectors $\chi_{\dot{\alpha}}$ with transformation law (2.40) are called *right Weyl spinors*. The indices $\dot{\alpha},\dot{\beta}$ are called the spinor ones.

Analogously to previous discussion, one introduces the matrices

$$\varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.41}$$

such that

$$\varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}{}^{\dot\gamma}. \tag{2.42}$$

The matrices (2.41) are used to rise and lower the indices $\dot\alpha$, $\dot\beta$ of right Weyl spinors

$$\chi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\chi_{\dot\beta}, \qquad \chi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\chi^{\dot\beta}. \tag{2.43}$$

For the right Weyl spinors one can prove the identity analogous to (2.39),

$$\chi_1'^{\dot\alpha}\chi'_{2\dot\alpha} = \chi_1^{\dot\alpha}\chi_{2\dot\alpha}, \tag{2.44}$$

where $\chi'_{\dot\alpha}$ is given by (2.40). The relation (2.44) means that the expression $\chi_1^{\dot\alpha}\chi_{2\dot\alpha}$ is Lorentz invariant.

Sometimes, the spinors $\bar\chi_{\dot\alpha}$ are called *dotted* spinors while the spinors φ^α are called *undotted* ones.

Since the dotted spinors transform with the help of conjugated matrix N^*, we can define the operation of conjugation for the spinors by the rule

$$(\varphi_\alpha)^* = \bar\varphi_{\dot\alpha}. \tag{2.45}$$

where $\bar\varphi_{\dot\alpha}$ is some dotted spinor.

Note also the following important properties for the left and right Weyl spinors

$$\begin{aligned}(\varphi_1\varphi_2) &\equiv \varphi_1^\alpha\varphi_{2\alpha} = -\varphi_2^\alpha\varphi_{1\alpha} = -(\varphi_2\varphi_1),\\ (\chi_1\chi_2) &\equiv \chi_{1\dot\alpha}\chi_2^{\dot\alpha} = -\chi_{2\dot\alpha}\chi_1^{\dot\alpha} = -(\chi_2\chi_1).\end{aligned} \tag{2.46}$$

Let us consider the matrices σ_m given by Eq. (2.20). Their matrix elements are denoted as $(\sigma_m)_{\alpha\dot\alpha}$. We introduce also the sigma-matrices with upper spinor indices

$$(\sigma_m)^{\alpha\dot{\alpha}} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}(\sigma_m)_{\beta\dot{\beta}}. \tag{2.47}$$

Using the explicit form of the matrices σ_m, (2.20), and $\varepsilon^{\alpha\beta}$, $\varepsilon^{\dot{\alpha}\dot{\beta}}$, (2.36), (2.41), one can show that

$$\tilde{\sigma}_m = (\sigma_0, -\vec{\sigma}). \tag{2.48}$$

The matrices σ_m and $\tilde{\sigma}_m$ possess many useful identities, for example

$$(\sigma_m\tilde{\sigma}_n + \sigma_n\tilde{\sigma}_m)_\alpha{}^\beta = -2\eta_{mn}\delta_\alpha{}^\beta, \tag{2.49a}$$
$$(\tilde{\sigma}_m\sigma_n + \tilde{\sigma}_n\sigma_m)^{\dot{\alpha}}{}_{\dot{\beta}} = -2\eta_{mn}\delta^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{2.49b}$$
$$\mathrm{tr}(\sigma_m\tilde{\sigma}_n) = -2\eta_{mn}, \tag{2.49c}$$
$$(\sigma^m)_{\alpha\dot{\alpha}}(\tilde{\sigma}_m)^{\dot{\beta}\beta} = -2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}}, \tag{2.49d}$$
$$\sigma_m = N\sigma_n N^+\big(\Lambda^{-1}(N)\big)^n{}_m. \tag{2.49e}$$

Here $\Lambda(N)$ is given by eq (2.27). Eq. (2.49e) means that the matrices σ_m are invariant tensors of $\mathrm{SL}(2,\mathbb{C})$ group.

Let us consider the following expressions:

$$\begin{aligned} v^m &= (\bar{\chi}\tilde{\sigma}^m\varphi) \equiv \bar{\chi}_{\dot{\alpha}}(\tilde{\sigma}^m)^{\dot{\alpha}\alpha}\varphi_\alpha, \\ u_m &= (\varphi\sigma_m\bar{\chi}) \equiv \varphi^\alpha(\sigma_m)_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}}. \end{aligned} \tag{2.50}$$

Using the transformation laws for the spinors φ_α, $\bar{\chi}_{\dot{\alpha}}$, Eqs. (2.34), (2.40) and (2.49e), one can show that v^m is a contravariant vector while u_m is covariant one.

2.4 Lie algebra of the Poincare group

Here we will consider some notions of the Lie algebra of the Poincare group on the base of one special example. One can show that the results, we are interesting, do not depend on the example we begin with.

The infinitesimal coordinate transformations corresponding to the inhomogeneous Lorentz transformations have been written in sect. 2.1 in the form

$$x'^m = x^m + \omega^m{}_n x^n + a^m = x^m + \delta x^m, \tag{2.51}$$

where $\omega^m{}_n$ and a^m are the infinitesimal parameters, $\omega_{mn} = -\omega_{nm}$. Let $t^m(x)$ be a vector field which is determined by the following transformation law under the inhomogeneous Lorentz transformations

$$t'^m(x') = \frac{\partial x'^m}{\partial x^n} t^n(x), \tag{2.52}$$

where x'^m is given by Eq. (2.51). The equation (2.52) can be rewritten as

$$t'^m(x + \delta x) = \Lambda^m{}_n t^n(x), \tag{2.53}$$

where $\Lambda^m{}_n = \delta^m{}_n + \omega^m{}_n$ is the Lorentz matrix. Therefore

$$t'^m(x + \omega x + a) = (\delta^m{}_n + \omega^m{}_n) t^n(x), \tag{2.54}$$

or, denoting $\delta t^m(x) = t'^m(x) - t^m(x)$, one gets

$$\delta t^m(x) = a^n \partial_n t^m(x) + \omega^m{}_n t^n(x) - \omega^n{}_k x^k \partial_n t^m(x). \tag{2.55}$$

The relation (2.55) can be written as follows

$$\delta t^m(x) = -ia^r(P_r)^m{}_n t^n(x) + \frac{i}{2}\omega^{rs}(J_{rs})^m{}_n t^n(x), \qquad (2.56)$$

where

$$\begin{aligned}
(P_r)^m{}_n &= \delta^m{}_n(-i\partial_r), \\
(J_{rs})^m{}_n &= \eta_{rk}x^k(P_s)^m{}_n - \eta_{sk}x^k(P_r)^m{}_n + (M_{rs})^m{}_n, \\
(M_{rs})^m{}_n &= i(\delta^m{}_s\eta_{rn} - \delta^m{}_r\eta_{sn}).
\end{aligned} \qquad (2.57)$$

Using the explicit form of operators P_r, J_{rs}, (2.57), we calculate the commutation relations among these operators. The result is written as follows

$$\begin{aligned}
[P_r, P_s] &= 0, \\
[J_{rs}, P_m] &= i(\eta_{rm}P_s - \eta_{sm}P_r), \\
[J_{mn}, J_{rs}] &= i(\eta_{mr}J_{ns} - \eta_{ms}J_{nr} + \eta_{ns}J_{mr} - \eta_{nr}J_{ms}).
\end{aligned} \qquad (2.58)$$

The expressions (2.57) are called the *generators of Poincare group in vector representation.* The relations (2.58) form the commutation relations of these generators. Sometimes, the relations (2.58) themselves are called the Poincare algebra.

One can show that the infinitesimal transformations of any tensor or spinor field under inhomogeneous Lorentz transformations (2.51) can be written in the form analogous to (2.56),

$$\delta\Phi(x) = -ia^m P_m\Phi(x) + \frac{i}{2}\omega^{mn}J_{mn}\Phi(x), \qquad (2.59)$$

with some generators P_m and J_{mn}. The operators P_m are called the *translation generators* and J_{mn} are called the

Lorentz rotation generators. The commutation relations among the operators P_m, J_{mn} also have the form (2.58), independently of the type of fields Φ in (2.59). We would like to emphasize that the algebra (2.58) is a mathematical expression of the special relativity symmetry.

3 Superspace and superfields

3.1 Supersymmetry algebra

Here we will discuss how to extend the Poincare algebra (2.58) by means of new generators which can provide symmetry between bosons and fermions. A general idea of such an extension is based on the use of spinor generators Q^I_α and $\bar{Q}^I_{\dot\alpha}$ with dotted and undotted indices α, $\dot\alpha$ and $I = 1, 2, \ldots, \mathcal{N}$. The integer $\mathcal{N}$ numerates a number of new generators.

It is postulated that the generators Q^I_α, $\bar{Q}^I_{\dot\alpha}$ possess a fermionic nature and their commutation relations are given in terms of anticommutators. We want to preserve the algebra (2.58) for P_m and J_{mn} and should find, additionally, the following commutators and anticommutators

$$[P_m, Q^I_\alpha], \qquad [P_m, \bar{Q}^I_{\dot\alpha}], \tag{3.1a}$$

$$[J_{mn}, Q^I_\alpha], \qquad [J_{mn}, \bar{Q}^I_{\dot\alpha}], \tag{3.1b}$$

$$\{Q^I_\alpha, Q^J_\beta\}, \qquad \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\}, \tag{3.1c}$$

$$\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\}, \tag{3.1d}$$

where $\{A, B\} = AB + BA$ is the anticommutator. A natural assumption is that the right hand sides of above commutators and anticommutators should be linear combinations of all generators P_m, J_{mn}, Q^I_α, $\bar{Q}^I_{\dot\alpha}$ with some coefficients. The Lorentz invariance imposes the conditions on these coefficients: they can be constructed only from the invariant tensors of Lorentz and SL$(2, \mathbb{C})$, i.e., η_{mn}, $\varepsilon_{\alpha\beta}$, $\varepsilon_{\dot\alpha\dot\beta}$, $(\sigma_m)_{\alpha\dot\alpha}$, $(\tilde\sigma_m)^{\dot\alpha\alpha}$. Therefore, the most general forms of the commutators and anticommutators (3.1) are written as follows:

$$
\begin{aligned}
[P_m, Q^I_\alpha] &= c_1(\sigma_m)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha I}, \\
[P_m, \bar{Q}^I_{\dot\alpha}] &= c_2(\tilde\sigma_m)_{\dot\alpha\alpha}Q^{\alpha I}, \\
[J_{mn}, Q^I_\alpha] &= c_3(\sigma_{mn})_\alpha{}^\beta Q^I_\beta, \\
[J_{mn}, \bar{Q}^I_{\dot\alpha}] &= c_4(\tilde\sigma_{mn})_{\dot\alpha}{}^{\dot\beta}\bar{Q}^I_{\dot\beta}, \\
\{Q^I_\alpha, Q^J_\beta\} &= c_5\varepsilon_{\alpha\beta}Z^{IJ} + \tilde{c}_5(\sigma^{mn})_{\alpha\beta}J_{mn}X^{IJ}, \\
\{\bar{Q}^I_{\dot\alpha}, \bar{Q\beta}^J\} &= c_6\varepsilon_{\alpha\beta}\bar{Z}^{IJ} + \tilde{c}_6(\tilde\sigma^{mn})_{\dot\alpha\dot\beta}J_{mn}\bar{X}^{IJ}, \\
\{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} &= c_7 2(\sigma^m)_{\alpha\dot\alpha}P^m\delta^{IJ},
\end{aligned}
\tag{3.2}
$$

where $c_1, \ldots, c_7$ are some numerical coefficients and $Z^{IJ} = -Z^{JI}$, $\bar{Z}^{IJ} = -\bar{Z}^{JI}$, $X^{IJ} = X^{JI}$, $\bar{X}^{IJ} = \bar{X}^{JI}$ are some matrices. Here

$$
\begin{aligned}
(\sigma_{mn})_{\alpha\beta} = -\frac{1}{4}(\sigma_m\tilde\sigma_n - \sigma_n\tilde\sigma_m)_{\alpha\beta}, \\
(\tilde\sigma_{mn})_{\dot\alpha\dot\beta} = -\frac{1}{4}(\tilde\sigma_m\sigma_n - \tilde\sigma_n\sigma_m)_{\dot\alpha\dot\beta}.
\end{aligned}
\tag{3.3}
$$

Of course, the coefficients c_1, c_2, c_3, c_4 are written down without any calculations since they are defined by the transformation laws of the dotted and undotted spinors under inhomogeneous Lorentz group. We do not write yet these coefficients to emphasize a general idea how all coefficients in (3.3) can be found .

To fix the coefficients $c_1, \ldots, c_7$ in the algebra (eq2.2) one uses the Jacobi identities which are written in terms of double commutators and anticommutators. We will use the standard terminology, when the operators P_m, J_{mn} are called bosonic and $Q^I_\alpha, \bar{Q}^I_{\dot{\alpha}}$ are called fermionic. Let B is some bosonic operator while F is some fermionic one. The Jacobi identities for such operators have the form

$$\begin{aligned}
[B_1,[B_2,B_3]]+[B_2,[B_3,B_1]]+[B_3,[B_1,B_2]] &= 0, \\
[B,\{F_1,F_2\}]+\{F_1,[F_2,B]\}-\{F_2,[B,F_1]\} &= 0, \\
[B_1,[B_2,F]]+[B_2,[F,B_1]]+[F_1,[B_1,B_2]] &= 0, \\
[F_1,\{F_2,F_3\}]+[F_2,\{F_3,F_1\}]+[F_3,\{F_1,F_2\}] &= 0. \quad (3.4)
\end{aligned}$$

Substituting the generators P_m, J_{mn} instead of B and generators Q^I_α, $\bar{Q}^I_{\dot{\alpha}}$ instead of F into the identities (3.4) and using the relations (3.2), one can find all numerical coefficients in Eqs. (3.2). The results are written as

$$\begin{aligned}
c_1 &= c_2 = \tilde{c}_5 = \tilde{c}_6 = 0, \quad c_3 = c_4 = i, \quad c_5 = c_6 = c_7 = 1, \\
X^{IJ} &= \bar{X}^{IJ} = 0, \qquad (3.5)
\end{aligned}$$

and the operators Z^{IJ}, $\bar{Z}^{IJ}$ must commute with all generators.

Thus, one gets the algebra (3.2) in the final form

$$
\begin{aligned}
[P_m, Q^I_\alpha] &= 0, \qquad & [P_m, \bar{Q}^I_{\dot\alpha}] &= 0,\\
[J_{mn}, Q^I_\alpha] &= i(\sigma_{mn})_\alpha{}^\beta Q^I_\beta, \qquad & [J_{mn}, \bar{Q}^I_{\dot\alpha}] &= i(\tilde\sigma_{mn})_{\dot\alpha}{}^{\dot\beta} \bar{Q}^I_{\dot\beta},\\
\{Q^I_\alpha, Q^J_\beta\} &= \varepsilon_{\alpha\beta} Z^{IJ}, \qquad & \{\bar{Q}^I_{\dot\alpha}, \bar{Q}^J_{\dot\beta}\} &= \varepsilon_{\dot\alpha\dot\beta} \bar{Z}^{IJ},\\
& \{Q^I_\alpha, \bar{Q}^J_{\dot\alpha}\} = 2(\sigma^m)_{\alpha\dot\alpha} P_m \delta^{IJ}. & & \qquad (3.6)
\end{aligned}
$$

The equations (3.6), together with Eqs. (2.58), form the so called *Poincare superalgebra*. The fermionic generators Q^I_α, $\bar{Q}^I_{\dot\alpha}$ are called the *supercharges*, the generators Z, $\bar{Z}$ are called the *central charges*. If $\mathcal{N} = 1$, the supersymmetry is called *simple*, or $\mathcal{N} = 1$ supersymmetry. In this case all central charges are absent. If $\mathcal{N} > 1$, the supersymmetry is called *extended*, or $\mathcal{N}$-extended supersymmetry. The statement that the relations (3.6) represent the most general extension of the Poincare algebra (2.58) by means of fermionic generators is referred as the Haag - Lopuszanski- Sohnius theorem which was proved in 1975.

Further we will consider only $\mathcal{N} = 1$ supersymmetry.

3.2 Anticommuting variables

Supersymmetric field theories (as well as all fermionic field theories) are formulated naturally with the help of a notion of Grassmann algebra which operates with the so called anticommuting variables, or anticommuting numbers. Here we discuss only some simple properties of anticommuting variables.

Why do we have to think about anticommuting variables in the case under consideration? We know that the generators of the Poincare algebra (2.58) can be associated with the coordinate transformations of the Minkowski space (see Eqs. (2.51)–(2.57)). Therefore, it is naturally to expect that the generators of supersymmetry algebra can also be associated with the coordinate transformations of some space. Since the generators P_m, J_{mn} are included into the supersymmetry algebra (3.6), the coordinates x^m of the Minkowski space should also be included into this new space. As the supersymmetry algebra contains the supercharges Q_α, $\bar{Q}_{\dot{\alpha}}$ satisfying the anticommutation relations (see Eqs. (3.6)), it is natural to assume that these supercharges are also associated with some new coordinates. Since the supercharges satisfy the anticommutation relations, one can assume that the corresponding coordinates should be anticommuting as well. Moreover, since the supercharges are the undotted and dotted spinors, it is natural to assume that these new anticommuting coordinates should also be the undotted and dotted spinors.

Further we will follow the pragmatic point of view: we will avoid any rigorous definitions and describe only the rules of operation with the anticommuting variables. In principle, all necessary material concerning the anticommuting variables is given in standard texts on quantum filed theory, particularly in the sections concerning the Lagrangians for spinor fields and anticommuting ghost fields in quantum theory of gauge fields. Here we assume that the readers are familiar with all these notions.

We denote anticommuting variables as θ_α, $\bar\theta_{\dot\alpha}$. It means that θ_α is a left Weyl spinor and $\bar\theta_{\dot\alpha}$ is a right Weyl spinor. One can raise the spinor indices with the ε-tensors as usual:

$$\theta^\alpha = \varepsilon^{\alpha\beta}\theta_\beta, \qquad \bar\theta^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\theta_{\dot\beta}. \tag{3.7}$$

The basic relations expressing the fundamental property of the anticommuting variables have the form

$$\begin{aligned} \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} + \bar\theta_{\dot\beta}\bar\theta_{\dot\alpha} &= 0, \\ \theta_\alpha\theta_\beta + \theta_\beta\theta_\alpha &= 0, \\ \theta_\alpha\bar\theta_{\dot\alpha} + \bar\theta_{\dot\alpha}\theta_\alpha &= 0. \end{aligned} \tag{3.8}$$

These relations imply

$$\begin{aligned} (\theta_\alpha)^2 &= 0, \qquad \alpha = 1, 2, \\ (\bar\theta_{\dot\alpha})^2 &= 0, \qquad \dot\alpha = \dot1, \dot2, \\ \theta_\alpha\theta_\beta\theta_\gamma &= 0, \qquad \alpha, \beta, \gamma = 1, 2, \\ \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}\bar\theta_{\dot\gamma} &= 0, \qquad \dot\alpha, \dot\beta, \dot\gamma = \dot1, \dot2. \end{aligned} \tag{3.9}$$

The basic relations (3.8) allow us to simplify the expressions $\theta_\alpha\theta_\beta$ and $\bar\theta_{\dot\alpha}\bar\theta_{\dot\beta}$. For example, let us consider $\theta_\alpha\theta_\beta$ which is an antisymmetric 2×2 matrix. Any such a matrix is always proportional to the matrix $\varepsilon_{\alpha\beta}$ with some coefficient, say C, $\theta_\alpha\theta_\beta = C\varepsilon_{\alpha\beta}$. Contracting the spinor indices, one gets $\varepsilon^{\alpha\beta}\theta_\alpha\theta_\beta = C\varepsilon^{\beta\alpha}\varepsilon_{\alpha\beta} = 2C$ or $C = \frac{1}{2}\varepsilon^{\beta\alpha}\theta_\alpha\theta_\beta$. The similar considerations are applicable to the expression $\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}$. As a result,

$$\begin{aligned} \theta_\alpha\theta_\beta &= \frac{1}{2}\varepsilon_{\alpha\beta}\theta^2, & \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}} &= -\frac{1}{2}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^2, \\ \theta^\alpha\theta^\beta &= -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2, & \bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} &= \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\theta}^2, \end{aligned} \tag{3.10}$$

where

$$\theta^2 \equiv \theta^\alpha\theta_\alpha = \varepsilon^{\alpha\beta}\theta_\beta\theta_\alpha, \qquad \bar{\theta}^2 \equiv \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\alpha}}. \tag{3.11}$$

Due to the relations (3.8), (3.10), any function of anticommuting variables can be only a polynomial

$$\begin{aligned} f(\theta,\bar{\theta}) &= a + b^\alpha\theta_\alpha + \tilde{b}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} + c_{\alpha\dot{\alpha}}\theta^\alpha\bar{\theta}^{\dot{\alpha}} + g\,\theta^2 + \tilde{g}\,\bar{\theta}^2 \\ &+ f^\alpha\theta_\alpha\bar{\theta}^2 + \tilde{f}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\theta^2 + d\,\theta^2\bar{\theta}^2, \end{aligned} \tag{3.12}$$

where a, b^α, $\tilde{b}_{\dot{\alpha}}$ $c_{\alpha\dot{\alpha}}$, d, $\tilde{d}$, f_α, $\tilde{f}_{\dot{\alpha}}$, g are real or complex numbers. Sometimes, the expansion (3.12) is called a superfunction and the numbers a, b^α, $\tilde{b}_{\dot{\alpha}}$ $c_{\alpha\dot{\alpha}}$, d, $\tilde{d}$, f_α, $\tilde{f}_{\dot{\alpha}}$, g are called the components of the superfunction.

Now we introduce the notions of differentiation and the integration of the superfunction (3.12) over the Grassmann variables θ_α, $\bar{\theta}_{\dot{\alpha}}$. The derivatives with respect to anticommuting variables are denoted as

$$\frac{\partial}{\partial\theta^\alpha} \equiv \partial_\alpha, \qquad \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \equiv \bar{\partial}_{\dot{\alpha}} \tag{3.13}$$

and satisfy the following four rules:

i) The derivative is a linear operation;

ii)

$$\partial_\alpha \theta_\beta = \delta_\alpha{}^\beta, \qquad \bar{\partial}_{\dot{\alpha}} \bar{\theta}^{\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}. \tag{3.14}$$

iii)

$$\begin{aligned}\partial_\alpha(\theta^\beta\theta^\gamma) &= \delta_\alpha{}^\beta\theta^\gamma - \delta_\alpha{}^\gamma\theta^\beta,\\ \bar{\partial}_{\dot{\alpha}}(\bar{\theta}^{\dot{\beta}}\bar{\theta}^{\dot{\gamma}}) &= \delta_{\dot{\alpha}}{}^{\dot{\beta}}\bar{\theta}^{\dot{\gamma}} - \delta_{\dot{\alpha}}{}^{\dot{\gamma}}\bar{\theta}^{\dot{\beta}}.\end{aligned} \tag{3.15}$$

iv)

$$\begin{aligned}\bar{\partial}_{\dot{\alpha}}\theta^\beta(...) &= -\theta^\beta\bar{\partial}_{\dot{\alpha}}(...),\\ \partial_\alpha\bar{\theta}^{\dot{\beta}}(...) &= -\bar{\theta}^{\dot{\beta}}\partial_\alpha(...).\end{aligned} \tag{3.16}$$

Using these rules i) – iv), one can show that the Grassmann derivatives are anticommuting

$$\begin{aligned}\partial_\alpha\partial_\beta + \partial_\beta\partial_\alpha &= 0,\\ \bar{\partial}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}} + \bar{\partial}_{\dot{\beta}}\bar{\partial}_{\dot{\alpha}} &= 0,\\ \partial_\alpha\bar{\partial}_{\dot{\beta}} + \bar{\partial}_{\dot{\beta}}\partial_\alpha &= 0.\end{aligned} \tag{3.17}$$

In particular,

$$\partial_\alpha^2 = \bar{\partial}_{\dot{\alpha}}^2 = 0, \qquad \alpha = 1, 2, \quad \dot{\alpha} = \dot{1}, \dot{2}. \tag{3.18}$$

In principle, there are two types of derivatives with respect to anticommuting variables, that is left $\overrightarrow{\partial_\alpha}$ and right $\overleftarrow{\partial_\alpha}$ ones. Here we consider only left derivatives $\overrightarrow{\partial_\alpha} \equiv \partial_\alpha$ since only they are sufficient for our purposes.

Using the relations (3.17) one can show, analogously to (3.10), that the expressions $\partial_\alpha \partial_\beta$, $\bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}}$ can be simplified as follows

$$\begin{aligned} \partial_\alpha \partial_\beta &= \frac{1}{2} \varepsilon_{\alpha\beta} \partial^2, & \bar{\partial}_{\dot{\alpha}} \bar{\partial}_{\dot{\beta}} &= -\frac{1}{2} \varepsilon_{\dot{\alpha}\dot{\beta}} \bar{\partial}^2, \\ \partial^\alpha \partial^\beta &= -\frac{1}{2} \varepsilon^{\alpha\beta} \partial^2, & \bar{\partial}^{\dot{\alpha}} \bar{\partial}^{\dot{\beta}} &= \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\partial}^2, \end{aligned} \tag{3.19}$$

where

$$\partial^2 \equiv \partial^\alpha \partial_\alpha, \qquad \bar{\partial}^2 \equiv \bar{\partial}_{\dot{\alpha}} \bar{\partial}^{\dot{\alpha}}. \tag{3.20}$$

Let us consider now the integration over anticommuting variables. Since $(\partial_\alpha)^2 = 0$, $(\bar{\partial}_{\dot{\alpha}})^2 = 0$ (no contraction over α or $\dot{\alpha}$), an integration can not be defined as an operation inverse to differentiation. A correct definition of the integration over anticommuting variables has been given by F. Berezin. The Berezin integral is based on the following rules:

i) Integration is a linear operation.

ii) Let $d\theta_\alpha$, $d\bar{\theta}^{\dot{\alpha}}$ are the extra parameters anticommuting among themselves and with all $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$ variables. Then, by definition,

$$\int d\theta_\alpha F = \partial_\alpha F, \qquad \int d\bar{\theta}^{\dot{\alpha}} G = \varepsilon^{\dot{\alpha}\dot{\beta}} \partial_{\dot{\beta}} G. \tag{3.21}$$

iii) Multiple integration is defined as repeated.

According to these rules, one has

$$\int d\theta_\alpha \theta^\beta = \delta_\alpha{}^\beta, \qquad \int d\bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\beta}} = \delta^{\dot{\alpha}}{}_{\dot{\beta}}. \tag{3.22}$$

Let us define

$$d^2\theta = \frac{1}{4}\varepsilon^{\alpha\beta}d\theta_\alpha d\theta_\beta, \qquad d^2\bar\theta = \frac{1}{4}\varepsilon_{\dot\alpha\dot\beta}d\bar\theta^{\dot\alpha}d\bar\theta^{\dot\beta}, \tag{3.23}$$

then

$$\int d^2\theta\theta^2 = 1, \qquad \int d^2\bar\theta\bar\theta^2 = 1. \tag{3.24}$$

Using the rules (3.21) and Eqs. (3.23), it is easy to see that

$$\int d^2\theta\partial_\alpha F = 0, \qquad \int d^2\bar\theta\bar\partial_{\dot\alpha}G = 0, \tag{3.25}$$

for arbitrary superfunctions F and G.

The relations (3.24) have a clear interpretation. Let us consider

$$\begin{aligned}\int d^2\theta\theta^2 F(\theta) &= \int d^2\theta\theta^2(a + b_\alpha\theta^\alpha + d\,\theta^2) \\ &= \int d^2\theta\theta^2 a = a\int d^2\theta\theta^2 = a = F(0).\end{aligned} \tag{3.26}$$

Here we have used the property $\theta_\alpha\theta_\beta\theta_\gamma = 0$. Note that Eq. (3.26) is analogous to the well known property of usual delta-function $\int_{-\infty}^{\infty}dx\delta(x)f(x) = f(0)$. Therefore one can treat θ^2, $\bar\theta^2$ as the delta-functions of the Grassmann variables

$$\delta^2(\theta) = \theta^2, \qquad \delta^2(\bar\theta) = \bar\theta^2. \tag{3.27}$$

As a consequence, the following properties are satisfied

$$\begin{aligned}&\int d^2\theta\,\delta^2(\theta) = 1, &&\int d^2\bar\theta\,\delta^2(\bar\theta) = 1, \\ &\int d^2\theta\,\delta^2(\theta)F(\theta) = F(0), &&\int d^2\bar\theta\,\delta^2(\bar\theta)F(\bar\theta) = F(0).\end{aligned} \tag{3.28}$$

Note also that the Grassmann delta-functions are not singular at zero

$$\delta^2(\theta)|_{\theta=0} = 0, \qquad \delta^2(\bar\theta)|_{\bar\theta=0} = 0. \tag{3.29}$$

Therefore, all Grassmann loop integrations in the quantum computations in supersymmetric theories are always convergent while all so-called tadpole contributions simply vanish due to the property (3.29).

It is convenient to define the expressions

$$\delta^4(\theta) = \delta^2(\theta)\delta^2(\bar\theta), \qquad d^4\theta = d^2\theta d^2\bar\theta. \tag{3.30}$$

Then, the following equality takes place

$$\int d^4\theta\, \delta^4(\theta) F(\theta, \bar\theta) = F(\theta, \bar\theta)|_{\theta=0, \bar\theta=0} \tag{3.31}$$

for arbitrary superfunction $F(\theta, \bar\theta)$.

3.3 Superspace

Let θ_α, $\bar\theta_{\dot\alpha}$ be the anticommuting variables. We assume that they are conjugated to each other,

$$(\theta_\alpha)^* = \bar\theta_{\dot\alpha}, \qquad (\bar\theta_{\dot\alpha})^* = \theta_\alpha, \tag{3.32}$$

where "$*$" means the complex conjugation. We define also the conjugation rule for the products of anticommuting variables as follows:

$$\begin{aligned} &(\theta_\alpha\theta_\beta)^* = \bar\theta_{\dot\beta}\bar\theta_{\dot\alpha}, \qquad (\bar\theta_{\dot\beta}\bar\theta_{\dot\alpha})^* = \theta_\alpha\theta_\beta, \\ &(\theta_\alpha\bar\theta_{\dot\beta})^* = \theta_\beta\bar\theta_{\dot\alpha}, \qquad (\theta\sigma^m\bar\theta)^* = (\theta\sigma^m\bar\theta). \end{aligned} \tag{3.33}$$

Superspace is defined as a manifold parameterized by the variables $\{x^m, \theta_\alpha, \bar{\theta}_{\dot{\alpha}}\}$, were x^m are the coordinates of Minkowski space and θ_α, $\bar{\theta}_{\dot{\alpha}}$ are anticommuting variables. Usually, x^m are called the *bosonic coordinates* while θ_α, $\bar{\theta}_{\dot{\alpha}}$ are called *fermionic coordinates.* The dimension of such superspace is equal to eight: four bosonic dimensions and four fermionic ones.

Any function defined on superspace is called a *superfield*, $V = V(x, \theta, \bar{\theta})$. Taking into account that any superfunction is a polynomial of anticommuting variables, one can write

$$\begin{aligned} V(x,\theta,\overline{\theta}) \;\; = \;\; & A(x) + \theta^\alpha\psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}}\bar{\phi}^{\dot{\alpha}}(x) + \theta^2 F(x) \\ & +\bar{\theta}^2 G(x) + (\theta\sigma^m\bar{\theta})A_m(x) + \bar{\theta}^2\theta^\alpha\lambda_\alpha(x) \\ & +\theta^2\overline{\theta}_{\dot{\alpha}}\overline{\eta}^{\dot{\alpha}}(x) + \theta^2\overline{\theta}^2 D(x). \end{aligned} \tag{3.34}$$

The coefficients $A(x)$, $\psi_\alpha(x)$, $\bar{\phi}^{\dot{\alpha}}(x)$, $F(x)$, $G(x)$, $A_m(x)$, $\lambda_\alpha(x)$, $\bar{\eta}^{\dot{\alpha}}(x)$, $D(x)$ in the decomposition (3.34) are called *component fields.* All component fields are the usual fields in Minkowski space. We see that a superfield automatically includes a lot of usual fields. We will consider further only the superfields which are the scalars under Lorentz rotations. In this case $A(x)$, $F(x)$, $G(x)$, $D(x)$ are the scalar fields, $A_m(x)$ is a vector field, $\psi_\alpha(x)$, $\lambda_\alpha(x)$ are left Weyl spinors and $\bar{\psi}^{\dot{\alpha}}(x)$, $\bar{\eta}^{\dot{\alpha}}(x)$ are right Weyl spinors.

One can impose a reality condition on the superfield (3.34)

$$V^* = V, \tag{3.35}$$

where the conjugation rules of anticommuting variables are given by Eqs. (3.32), (3.33). The condition (3.35) implies that

$A(x)$, $D(x)$ are real scalar fields, $A_m(x)$ is a real vector field and the following equalities take place

$$G = F^*, \quad \bar{\phi}_{\dot{\alpha}} = (\psi_\alpha)^* \equiv \bar{\psi}_{\dot{\alpha}}, \quad \bar{\eta}_{\dot{\alpha}} = (\lambda_\alpha)^* \equiv \bar{\lambda}_{\dot{\alpha}}. \tag{3.36}$$

As a result, the real scalar superfield is given by the following decomposition

$$\begin{aligned} V(x,\theta,\bar{\theta}) &= A(x) + \theta^\alpha \psi_\alpha(x) + \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}(x) + \theta^2 F(x) \\ &\quad + \bar{\theta}^2 F^*(x) + (\theta\sigma^m\bar{\theta}) A_m(x) + \bar{\theta}^2 \theta^\alpha \lambda_\alpha(x) \\ &\quad + \theta^2 \bar{\theta}_{\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}(x) + \theta^2 \bar{\theta}^2 D(x). \end{aligned} \tag{3.37}$$

Note that the real scalar superfield (3.37) has less number of independent component fields in comparison with the general one (3.34).

Our next purpose is to realize the supercharges Q_α, $\bar{Q}_{\dot{\alpha}}$ as the operators acting on superfields. We consider the superspace coordinate transformations of the form

$$\begin{aligned} x'^m &= x^m - i(\epsilon\sigma^m\bar{\theta} - \theta\sigma^m\bar{\epsilon}), \\ \theta'^\alpha &= \theta^\alpha + \epsilon^\alpha, \\ \bar{\theta}'_{\dot{\alpha}} &= \bar{\theta}_{\dot{\alpha}} + \bar{\epsilon}_{\dot{\alpha}}. \end{aligned} \tag{3.38}$$

The transformations (3.38) are called the *supertranslations* or *supersymmetry transformations*. Here ϵ^α, $\bar{\epsilon}_{\dot{\alpha}}$ are anticommuting transformation parameters. We assume that the infinitesimal transformations of any superfield $V = V(x,\theta,\bar{\theta})$ under supertranslations (3.38) should be written as

$$\delta V(x,\theta,\bar{\theta}) = i(\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}) V(x,\theta,\bar{\theta}), \tag{3.39}$$

where Q_α are $\bar{Q}^{\dot{\alpha}}$ are some operators satisfying the (anti)commutation relations of the superalgebra (3.6) for $\mathcal{N} = 1$. In particular, the following relations should take place

$$\{Q_\alpha, Q_\beta\} = 0, \qquad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0,$$
$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2(\sigma^m)_{\alpha\dot{\alpha}} P_m. \tag{3.40}$$

The superfield $V = V(x, \theta, \bar{\theta})$ is called *scalar superfield* if it transforms under supertranslations (3.38) as

$$V'(x', \theta', \bar{\theta}') = V(x, \theta, \bar{\theta}). \tag{3.41}$$

Substituting Eq. (3.38) into (3.41) one gets

$$V'(x + \delta x, \theta + \epsilon, \bar{\theta} + \bar{\epsilon}) = V(x, \theta, \bar{\theta}), \tag{3.42}$$

where $\delta x^m = -i(\epsilon\sigma^m\bar{\theta} - \theta\sigma^m\bar{\epsilon})$. Expanding Eq. (3.42) up to first order in the parameters ϵ, $\bar{\epsilon}$, one obtains

$$\begin{aligned}&V'(x, \theta, \bar{\theta}) + \delta x^m \partial_m V(x, \theta, \bar{\theta})\\ &+\epsilon^\alpha \partial_\alpha V(x, \theta, \bar{\theta}) + \bar{\epsilon}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} V(x, \theta, \bar{\theta}) = V(x, \theta, \bar{\theta}).\end{aligned} \tag{3.43}$$

Therefore,

$$\begin{aligned}\delta V(x, \theta, \bar{\theta}) &\equiv V'(x, \theta, \bar{\theta}) - V(x, \theta, \bar{\theta})\\ &= -\epsilon^\alpha \partial_\alpha V - \bar{\epsilon}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} V - \delta x^m \partial_m V\\ &= i(i\epsilon^\alpha \partial_\alpha V) - i(-i\bar{\epsilon}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}} V)\\ &\quad +i\epsilon^\alpha {\sigma^m}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_m V - i\theta^\alpha {\sigma^m}_{\alpha\dot{\alpha}} \bar{\epsilon}^{\dot{\alpha}} \partial_m V\\ &= i\epsilon^\alpha (i\partial_\alpha + {\sigma^m}_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_m) V\\ &\quad -i\bar{\epsilon}^{\dot{\alpha}} (-i\bar{\partial}_{\dot{\alpha}} - \theta^\alpha {\sigma^m}_{\alpha\dot{\alpha}} \partial_m) V.\end{aligned} \tag{3.44}$$

Comparing Eq. (3.44) with (3.39), we conclude that the supercharges are given by

$$Q_\alpha = i\partial_\alpha + \sigma^m{}_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m, \qquad \bar{Q}_{\dot{\alpha}} = -i\bar{\partial}_{\dot{\alpha}} - \theta^\alpha\sigma^m{}_{\alpha\dot{\alpha}}\partial_m. \tag{3.45}$$

Now it is easy to check that the operators (3.45) satisfy the relations (3.40), i.e., they realize the representation of the supersymmetry algebra (3.6) on scalar superfields.

The relations (3.39), (3.45) allow us to find the transformation laws of the component fields under the supertranslations (3.38). For example, we take the real scalar superfield $V(x,\theta,\bar{\theta})$ in the form (3.37), substitute it into Eq. (3.39) and apply (3.45):

$$\begin{aligned}
&\delta A + \theta^\alpha\delta\psi_\alpha + \bar{\theta}_{\dot{\alpha}}\delta\bar{\psi}^{\dot{\alpha}} + \theta^2\delta F + \bar{\theta}^2\delta F^* \\
&+\theta^\alpha\bar{\theta}^{\dot{\alpha}}\delta A_{\alpha\dot{\alpha}} + \bar{\theta}^2\theta^\alpha\delta\lambda_\alpha + \theta^2\bar{\theta}_{\dot{\alpha}}\delta\bar{\eta}^{\dot{\alpha}} + \theta^2\bar{\theta}^2\delta D \\
&= \big(i\epsilon^\alpha(i\partial_\alpha + \bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}) - i\bar{\epsilon}^{\dot{\alpha}}(-i\partial_{\dot{\alpha}} - \theta^\alpha\partial_{\alpha\dot{\alpha}})\big) \\
&\times(A + \theta^\beta\psi_\beta + \bar{\theta}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}} + \theta^2 F + \bar{\theta}^2 F^* + \theta^\beta\bar{\theta}^{\dot{\beta}}A_{\beta\dot{\beta}} \\
&+\theta^2\theta^\beta\lambda_\beta + \bar{\theta}^2\bar{\theta}_{\dot{\beta}}\bar{\lambda}^{\dot{\beta}} + \theta^2\bar{\theta}^2 D),
\end{aligned} \tag{3.46}$$

where

$$\partial_{\alpha\dot{\alpha}} = \sigma^m{}_{\alpha\dot{\alpha}}\partial_m, \qquad A_{\alpha\dot{\alpha}} = \sigma^m{}_{\alpha\dot{\alpha}}A_m. \tag{3.47}$$

Now, we have to compute all derivatives over the anticommuting variables in right hand side of Eq. (3.46) and take into account Eq. (3.10) as well as the identities $\theta_\alpha\theta_\beta\theta_\gamma = 0$, $\bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\bar{\theta}_{\dot{\gamma}} = 0$. Comparing the expressions at the corresponding powers of θ-variables, we arrive at the following set of equalities

$$
\begin{aligned}
\delta A &= -\epsilon^{\alpha}\psi_{\alpha} - \bar{\epsilon}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}, \\
\delta\psi_{\alpha} &= -2\epsilon_{\alpha}F - \bar{\epsilon}^{\dot{\alpha}}A_{\alpha\dot{\alpha}} - i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}A, \\
\delta F &= -\epsilon^{\alpha}\lambda_{\alpha}, \\
\delta A_{\alpha\dot{\alpha}} &= 2(\bar{\epsilon}_{\dot{\alpha}}\lambda_{\alpha} - \epsilon_{\alpha}\bar{\lambda}_{\dot{\alpha}}) - 2i(\epsilon_{\alpha}\partial_{\beta\dot{\alpha}}\psi^{\beta} + \bar{\epsilon}_{\dot{\alpha}}\partial_{\alpha\dot{\beta}}\bar{\psi}^{\dot{\beta}} \\
&\quad + i\partial_{\alpha\dot{\alpha}}(\bar{\epsilon}_{\dot{\beta}}\bar{\psi}^{\dot{\beta}}) - \epsilon^{\beta}\psi_{\beta}), \qquad (3.48) \\
\delta\lambda_{\alpha} &= -2\epsilon_{\alpha}D - i\epsilon_{\alpha}\partial_m A^m - 2i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}F, \\
\delta D &= -\frac{i}{2}\partial_{\alpha\dot{\alpha}}(\epsilon^{\alpha}\bar{\lambda}^{\dot{\alpha}} + \bar{\epsilon}^{\dot{\alpha}}\lambda^{\alpha}). \qquad (3.49)
\end{aligned}
$$

As we know from the course of quantum field theory, the scalar and vector fields describe the bosonic particles and they are called bosonic fields. The spinor fields describe the fermionic particles and are called fermionic fields. Equations (3.48) show explicitly that the supersymmetry transforms the bosonic fields into fermionic ones and vice versa.

In the conclusion of this section let us consider the question about the numbers of bosonic and fermionic components of a real superfield V which is given by Eq. (3.37). The bosonic fields A, F, D, A_m have 8 real components (two real scalar fields A, D, one four-component real vector A_m and a complex scalar F). The fermionic fields ψ_{α}, λ_{α} have also 8 real components (both of them have two complex components due to the spinor index $\alpha = 1, 2$). As a result, we see that the number of bosonic components of the superfield is equal to the number of its fermionic ones.

3.4 Supercovariant derivatives

Supercovariant derivatives D_α, $\bar{D}_{\dot{\alpha}}$ are defined by the following conditions

$$\delta D_\alpha V = D_\alpha i(\epsilon^\beta Q_\beta + \bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}})V = D_\alpha \delta V,$$
$$\delta \bar{D}_{\dot{\alpha}} V = \bar{D}_{\dot{\alpha}} i(\epsilon^\beta Q_\beta + \bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}})V = \bar{D}_{\dot{\alpha}} \delta V, \qquad (3.50)$$

which have to be valid for arbitrary superfield $V = V(x, \theta, \bar{\theta})$. The equations (3.50) mean that the expressions $D_\alpha V$, $\bar{D}_{\dot{\alpha}} V$ transform like V under the supertranslations (3.38). It follows from Eqs. (3.50) that

$$\begin{aligned} [D_\alpha, \epsilon^\beta Q_\beta + \bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}] &= 0, \\ [\bar{D}_{\dot{\alpha}}, \epsilon^\beta Q_\beta + \bar{\epsilon}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}] &= 0. \end{aligned} \qquad (3.51)$$

Since the parameters ϵ^β, $\bar{\epsilon}_{\dot{\beta}}$ are anticommuting, one gets from (3.51) the following identities

$$\begin{aligned} \{D_\alpha, Q_\beta\} &= 0, \qquad & \{D_\alpha, \bar{Q}_{\dot{\beta}}\} &= 0, \\ \{\bar{D}_{\dot{\alpha}}, Q_\alpha\} &= 0, \qquad & \{\bar{D}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} &= 0, \end{aligned} \qquad (3.52)$$

where the supercharges Q_α, $\bar{Q}_{\dot{\alpha}}$ are given by Eq. (3.45). The Eqs. (3.52) allow us to find the supercovariant derivatives explicitly. To do this we solve the equations (3.52) using the anzatz

$$D_\alpha = c_1\partial_\alpha + c_2\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \qquad \bar{D}_{\dot{\alpha}} = c_3\bar{\partial}_{\dot{\alpha}} + c_4\theta^\alpha\bar{\partial}_{\alpha\dot{\alpha}}, \qquad (3.53)$$

with some unknown coefficients c_1, c_2, c_3, c_4. Substituting Eqs. (3.53) into (3.52), one can find all coefficients [1]. The

[1] The coefficients c_1, c_3 are arbitrary while c_2, c_4 are expressed in terms of c_1, c_2. Here we choose $c_1 = 1$, $c_3 = -1$ for simplicity.

result looks like

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\sigma^m{}_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m = \partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}, \\ \bar D_{\dot\alpha} &= -\bar\partial_{\dot\alpha} - i\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m = -\bar\partial_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha}. \end{aligned} \tag{3.54}$$

Sometimes, the supercovariant derivatives are called the spinor derivatives. Note that the derivative ∂_m is covariant with respect to the supertranslations since the supercharges commute with the generators of space-time translations as it follows from the algebra (3.6).

The spinor derivatives (3.29) possess a number of important properties which are used in supersymmetric field theories. It is easy to check that they satisfy the algebra

$$\begin{gathered} \{D_\alpha, D_\beta\} = 0, \qquad \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0, \\ [D_\alpha, \partial_m] = 0, \qquad [\bar D_\alpha, \bar\partial_m] = 0, \\ \{D_\alpha, \bar D_{\dot\alpha}\} = -2i\partial_{\alpha\dot\alpha} = 2P_{\alpha\dot\alpha}. \end{gathered} \tag{3.55}$$

Since $D_\alpha D_\beta$ is antisymmetric 2×2 matrix we can repeat the same analysis as for $\theta_\alpha\theta_\beta$. It leads to the number of identities:

$$D_\alpha D_\beta = \frac{1}{2}\varepsilon_{\alpha\beta}D^2, \qquad \bar D_{\dot\alpha}\bar D_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar D^2,$$

$$D^\alpha D^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}D^2, \qquad \bar D^{\dot\alpha}\bar D^{\dot\beta} = -\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar D^2, \tag{3.56}$$

$$D_\alpha D_\beta D_\gamma = 0, \qquad \bar D_{\dot\alpha}\bar D_{\dot\beta}\bar D_{\dot\gamma} = 0, \tag{3.57}$$

where

$$\begin{aligned} D^2 &= D^\alpha D_\alpha, \qquad & \bar D^2 &= \bar D_{\dot\alpha}\bar D^{\dot\alpha}, \\ D^\alpha &= \varepsilon^{\alpha\beta}D_\beta, \qquad & \bar D^{\dot\alpha} &= \varepsilon^{\dot\alpha\dot\beta}\bar D_{\dot\beta}. \end{aligned} \tag{3.58}$$

Using the algebra of spinor derivatives (3.55) one can prove the identities

$$D^2\bar{D}_{\dot\alpha}D^2 = 0, \qquad \bar{D}^2D_\alpha\bar{D}^2 = 0, \tag{3.59a}$$

$$D^\alpha\bar{D}^2D_\alpha = \bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha}, \tag{3.59b}$$

$$D^2\bar{D}^2 + \bar{D}^2D^2 - 2D^\alpha\bar{D}^2D_\alpha = 16\Box, \tag{3.59c}$$

$$D^2\bar{D}^2D^2 = 16\Box D^2, \qquad \bar{D}^2D^2\bar{D}^2 = 16\Box\bar{D}^2, \tag{3.59d}$$

$$[D^2, \bar{D}_{\dot\alpha}] = -4i\partial_{\alpha\dot\alpha}D^\alpha, \qquad [\bar{D}^2, D_\alpha] = 4i\partial_{\alpha\dot\alpha}\bar{D}^{\dot\alpha}. \tag{3.59e}$$

3.5 Chiral superfields

Using the supercovariant derivatives D_α, $\bar{D}_{\dot\alpha}$ one can impose the constraints on the superfields which are consistent with the supersymmetry transformations. The simplest such a constraint is given by

$$\bar{D}_{\dot\alpha}\Phi(x,\theta,\bar\theta) = 0, \tag{3.60}$$

where $\Phi(x,\theta,\bar\theta)$ is a complex scalar superfield. Eq. (3.60) can be exactly solved. Let us act on both sides of Eq. (3.60) by the operator $e^{-i(\theta\sigma^m\bar\theta)\partial_m}$

$$e^{-i(\theta\sigma^m\bar\theta)\partial_m}\bar{D}_{\dot\alpha}e^{i(\theta\sigma^m\bar\theta)\partial_m}e^{-i(\theta\sigma^m\bar\theta)\partial_m}\Phi = 0. \tag{3.61}$$

Due to the identity

$$\begin{aligned} &e^{-i(\theta\sigma^m\bar\theta)\partial_m}\bar{D}_{\dot\alpha}e^{i(\theta\sigma^m\bar\theta)\partial_m} \\ &= e^{-i(\theta\sigma^m\bar\theta)\partial_m}(\bar\partial_{\dot\alpha} + i\theta^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m)e^{i(\theta\sigma^m\bar\theta)\partial_m} = -\bar\partial_{\dot\alpha}, \end{aligned} \tag{3.62}$$

Eq. (3.61) is simplified

$$\bar{\partial}_{\dot{\alpha}} \left(e^{-i(\theta\sigma^m\bar{\theta})\partial_m} \Phi(x, \theta, \bar{\theta}) \right) = 0. \tag{3.63}$$

Hence, the function $e^{-i(\theta\sigma^m\bar{\theta})\partial_m} \Phi(x, \theta, \bar{\theta})$ is independent on $\bar{\theta}_{\dot{\alpha}}$ variables, i.e.,

$$e^{-i(\theta\sigma^m\bar{\theta})\partial_m} \Phi(x, \theta, \bar{\theta}) = \Phi(x, \theta), \tag{3.64}$$

where $\Phi(x, \theta)$ is an arbitrary superfield depending on x^m and θ_α variables only. As a result, the solution of (3.65) is written as follows

$$\Phi(x, \theta, \bar{\theta}) = e^{i(\theta\sigma^m\bar{\theta})\partial_m} \Phi(x, \theta) = \Phi(x + i\theta\sigma^m\bar{\theta}, \theta). \tag{3.65}$$

A superfield satisfying the constraint (3.60) is called *chiral superfield.* Eq. (3.65) defines a general form of a chiral superfield. It is important to emphasize that the chiral superfield is obligatory complex.

Analogously, consider the constraint

$$D_\alpha \bar{\Phi}(x, \theta, \bar{\theta}) = 0. \tag{3.66}$$

A superfield satisfying the constraint (3.66) is called *antichiral.* One can show that a solution of (3.66) has the form

$$\bar{\Phi}(x, \theta, \bar{\theta}) = e^{-i(\theta\sigma^m\bar{\theta})\partial_m} \bar{\Phi}(x, \bar{\theta}) = \bar{\Phi}(x - i\theta\sigma^m\bar{\theta}, \bar{\theta}). \tag{3.67}$$

As a result, we see that the basic features of chiral and antichiral superfields are given by the special dependence on anticommuting variables: a chiral superfield depends essentially only on θ while an antichiral one depends essentially on $\bar{\theta}$ only.

Equations (3.65), (3.67) allow us to find the component structure of chiral and antichiral superfields. We begin with Eq. (3.65). Since $\Phi(x,\theta)$ is $\bar{\theta}$-independent, it is given by the following component decomposition

$$\Phi(x,\theta) = A(x) + \theta^\alpha \psi_\alpha(x) + \theta^2 F(x). \tag{3.68}$$

Hence

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) &= e^{i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x,\theta) \\
&= e^{i(\theta\sigma^m\bar{\theta})\partial_m}(A(x) + \theta^\alpha\psi_\alpha + \theta^2 F(x)) \\
&= A + \theta^\alpha\psi_\alpha + \theta^2 F + i(\theta\sigma^m\bar{\theta})\partial_m A \\
&\quad + i(\theta\sigma^m\bar{\theta})\theta^\alpha\partial_m\psi_\alpha - \frac{1}{2}(\theta\sigma^m\bar{\theta})(\theta\sigma^n\bar{\theta})\partial_m\partial_n A.
\end{aligned} \tag{3.69}$$

Now we have to use the identities (3.10) to simplify the following terms

$$\begin{aligned}
i(\theta\sigma^m\bar{\theta})\theta^\alpha\partial_m\psi_\alpha &= \frac{i}{2}(\theta^2\theta_{\dot{\alpha}}(\tilde{\sigma}^m)^{\dot{\alpha}\alpha}\partial_m\psi_\alpha), \\
-\frac{1}{2}(\theta\sigma^m\bar{\theta})(\theta\sigma^n\bar{\theta})\partial_m\partial_n A &= -\frac{1}{4}\theta^2\bar{\theta}^2\Box A.
\end{aligned} \tag{3.70}$$

Then, the component expansion of the chiral superfield is

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) &= A + \theta^\alpha\psi_\alpha + \theta^2 F + i(\theta\sigma^m\bar{\theta})\partial_m A \\
&\quad + \frac{i}{2}\theta^2\theta_{\dot{\alpha}}(\tilde{\sigma}^m)^{\dot{\alpha}\alpha}\partial_m\psi_\alpha - \frac{1}{4}\theta^2\bar{\theta}^2\Box A(x)
\end{aligned} \tag{3.71}$$

The component expansion for antichiral superfield can be obtained from (3.71) by conjugation.

Let us study the transformations of a chiral superfield under supertranslations. There is a general relation

$$\delta\Phi = i(\epsilon^{\alpha}Q_{\alpha} + \bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})\Phi, \tag{3.72}$$

which takes place for arbitrary superfield. Substituting the chiral superfield in the form (3.71) into (3.72), ones get

$$\delta A + i\theta^{\alpha}\delta\Psi_{\alpha} + \theta^2\delta F + ... = i(\epsilon^{\alpha}Q_{\alpha} - \bar{\epsilon}^{\dot{\alpha}}\bar{Q}_{\dot{\alpha}})(A + \theta^{\beta}\psi_{\beta} + \theta^2 F + ...), \tag{3.73}$$

where dots mean the other components which are expressed via A, ψ_{α}, F according to Eq. (3.71). Then we have to substitute the explicit forms of supercharges Q_{α}, $\bar{Q}_{\dot{\alpha}}$, (3.45), and compute the derivatives with respect to θ_{α}, $\bar{\theta}_{\dot{\alpha}}$. As a result, one gets

$$\begin{aligned}
\delta A(x) &= -\epsilon^{\alpha}\psi_{\alpha}(x), \\
\delta\psi_{\alpha}(x) &= -2\epsilon_{\alpha}F(x) - 2i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}A(x), \\
\delta F(x) &= -i\bar{\epsilon}_{\dot{\alpha}}(\tilde{\sigma}^m)^{\dot{\alpha}\alpha}\partial_m\psi_{\alpha}.
\end{aligned} \tag{3.74}$$

The same consideration for antichiral superfield gives

$$\begin{aligned}
\delta\bar{A}(x) &= -\bar{\epsilon}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(x), \\
\delta\bar{\psi}_{\dot{\alpha}}(x) &= -2\bar{\epsilon}_{\dot{\alpha}}\bar{F}(x) - 2i\epsilon^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{A}(x), \\
\delta\bar{F}(x) &= i\epsilon^{\alpha}(\sigma^m)_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}}.
\end{aligned} \tag{3.75}$$

In the conclusion of this section we point out that any function of a chiral superfield only is also a chiral superfield:

$$\bar{D}_{\dot{\alpha}}f(\Phi) = f'(\Phi)\bar{D}_{\dot{\alpha}}\Phi = 0. \tag{3.76}$$

Analogously, any function of antichiral superfield is also a antichiral superfield.

4 Superfield models

4.1 Superfield action

Any field model in quantum field theory is given by a set of fields and corresponding action functional depending on these fields. The action is usually written as an integral over the Minkowski space of a Lorenz invariant Lagrangian. We are going to develop a formulation of supersymmetric field theory, where a set of fields consists of superfields and the action functional is written as an integral over the superspace of a superfield Lagrangian.

We consider only the simplest models when the set of fields consists only of the real scalar superfield $V(x, \theta, \bar{\theta})$ and the chiral and antichiral superfields Φ, $\bar{\Phi}$.

First of all, let us introduce the notations.

i) The superspace coordinates are denoted as

$$z^M \equiv (x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}). \tag{4.1}$$

ii) The set of supercovariant derivatives is denoted as

$$D_M = (\partial_m, D_\alpha, \bar{D}_{\dot{\alpha}}). \tag{4.2}$$

iii) The measure of full $\mathcal{N} = 1$ superspace is denoted as

$$d^8z = d^4x d^2\theta d^2\bar{\theta} = d^4x d^4\theta, \tag{4.3}$$

while the measures on chiral and antichiral subspaces are

$$d^6z = d^4x d^2\theta, \qquad d^6\bar{z} = d^4x d^2\bar{\theta}, \tag{4.4}$$

respectively.

iv) The superspace delta-function is given by

$$\delta^8(z - z') = \delta^4(x - x')\delta^4(\theta - \theta'), \tag{4.5}$$

where

$$\delta^4(\theta-\theta') = \delta^2(\theta-\theta')\delta^2(\bar{\theta}-\bar{\theta}') = (\theta-\theta')^2(\bar{\theta}-\bar{\theta}')^2. \tag{4.6}$$

The most general action functional, which can be written as an integral over the superspace, has the form

$$S = \int d^8z \mathcal{L} + \int d^6z \mathcal{L}_c + \int d^6\bar{z} \bar{\mathcal{L}}_c, \tag{4.7}$$

where $\mathcal{L}$ is a real scalar superfield, $\mathcal{L}_c$ is a chiral scalar superfield and $\bar{\mathcal{L}}_c$ is an antichiral scalar one. The first term in Eq. (4.7) is given by the integral over full superspace, the second term represents the integral over chiral superspace and the last one is the integral over antichiral superspace.

One can ask a question, why not to write the following term $\int d^8z\mathcal{L}_c$, i.e., to integrate a chiral function $\mathcal{L}_c$ over full superspace? The answer sounds as follows: such an expression identically vanish. Indeed, using the properties of the integral over anticommuting variables, Eqs. (3.21)–(3.25), one writes

$$\int d^8z\mathcal{L}_c = \int d^4xd^2\theta\frac{1}{4}\varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\partial}^{\dot{\alpha}}\bar{\partial}^{\dot{\beta}}\mathcal{L}_c = \frac{1}{4}\int d^4xd^2\theta\bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}\mathcal{L}_c. \tag{4.8}$$

Note that the contraction of covariant spinor derivatives can be written as

$$\begin{aligned}\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}} &= (-\bar{\partial}_{\dot{\alpha}} - i\theta^{\alpha}\partial_{\alpha\dot{\alpha}})(\bar{\partial}^{\dot{\alpha}} - i\theta^{\alpha}\partial_{\alpha}{}^{\dot{\alpha}}) \\ &= -\bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}} + \text{total space-time derivatives.}\end{aligned} \tag{4.9}$$

Note also the similar relation for D_α derivatives

$$D^{\alpha}D_{\alpha} = -\partial^{\alpha}\partial_{\alpha} + \text{total space-time derivatives.} \tag{4.10}$$

Since total space-time derivatives can be discarded under the integration over d^4x , the expression (4.8) reads

$$\frac{1}{4}\int d^4xd^2\theta\bar{\partial}_{\dot{\alpha}}\bar{\partial}^{\dot{\alpha}}\mathcal{L}_c = \int d^4xd^2\theta(-\frac{1}{4})\bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}\mathcal{L}_c. \tag{4.11}$$

Recalling the definition of a chiral superfield, $\bar{D}_{\dot{\alpha}}\mathcal{L}_c = 0$, we conclude that

$$\int d^8z\mathcal{L}_c = 0. \tag{4.12}$$

Analogously

$$\int d^8z\bar{\mathcal{L}}_c = 0. \tag{4.13}$$

In principle, the integration over all anticommuting variables in the action (4.7) can be performed resulting to the action in the component form as an integral over Minkowski space. Indeed, since all integrals over anticommuting variables are nothing but the Grassmann derivatives, we have

$$S = \frac{1}{16}\int d^4x \partial^2\bar{\partial}^2\mathcal{L} + \frac{1}{4}\int d^4x\partial^2\mathcal{L}_c + \frac{1}{4}\int d^4x\bar{\partial}^2\bar{\mathcal{L}}_c. \tag{4.14}$$

Applying the relations (4.9), (4.10), one gets

$$S = \int d^4x \left(\frac{1}{16}D^2\bar{D}^2\mathcal{L} - \frac{1}{4}D^2\mathcal{L}_c - \frac{1}{4}\bar{D}^2\bar{\mathcal{L}}_c\right). \tag{4.15}$$

This relation is useful for finding a component form of superfield actions.

The main objects of classical field theory are the equations of motion. In the case under consideration the equations of motion can be written completely in superfield form. To find the superfield equations of motions we need the superfield identities generalizing the following useful relation in conventional field theory

$$\frac{\delta\phi(x)}{\delta\phi(x')} = \delta^4(x-x') \tag{4.16}$$

for arbitrary scalar field $\phi(x)$. In our case we have a real scalar superfield V, a chiral scalar superfield Φ and an antichiral one $\bar{\Phi}$. Therefore we have to calculate the following variational derivatives

$$\frac{\delta V(z)}{\delta V(z')}, \quad \frac{\delta\Phi(z)}{\delta\Phi(z')}, \quad \frac{\delta\bar{\Phi}(z)}{\delta\bar{\Phi}(z')}. \tag{4.17}$$

The first case $\delta V(z)/\delta V(z')$ is rather trivial. Indeed, due to the identity

$$\delta V(z) = \int d^8z'\delta^8(z-z')\delta V(z'), \tag{4.18}$$

we have

$$\frac{\delta V(z)}{\delta V(z')} = \delta^8(z-z'). \tag{4.19}$$

The calculation of the variational derivative $\delta\Phi(z)/\delta\Phi(z')$ is more complicated since $\Phi(z)$ is a constrained superfield, $\bar{D}_{\dot{\alpha}}\Phi = 0$. The $\Phi(z)$ and $\delta\Phi(z)$ are chiral superfields, therefore they should be integrated over the chiral superspace with the measure d^6z. Hence

$$\delta\Phi(z) = \int d^6z'\frac{\delta\Phi(z)}{\delta\Phi(z')}\delta\Phi(z') = \int d^8z'\delta^8(z-z')\delta\Phi(z'). \tag{4.20}$$

The last term in Eq. (4.20) can be transformed as follows

$$\begin{aligned}\int d^8z'\delta^8(z-z')\delta\Phi(z') &= -\frac{1}{4}\int d^6z'\bar{D}^2_{(z')}\delta^8(z-z')\delta\Phi(z')\\ &= \int d^6z'[-\frac{1}{4}\bar{D}^2_{(z)}\delta^8(z-z')]\delta\Phi(z'),\end{aligned} \tag{4.21}$$

where we apply the same tricks as in the equations (4.8)–(4.11). As a result, we find

$$\frac{\delta\Phi(z)}{\delta\Phi(z')} = -\frac{1}{4}\bar{D}^2\delta^8(z-z'). \tag{4.22}$$

Analogously, one ones get the last variational derivative in Eq. (4.17)

$$\frac{\delta\bar{\Phi}(z)}{\delta\bar{\Phi}(z')} = -\frac{1}{4}D^2\delta^8(z-z'). \tag{4.23}$$

Due to the properties of the spinor derivatives, (3.57), it is easy to see that the expressions (4.22) and (4.23) are chiral and antichiral superfields, respectively, with respect to both their argument z and z'.

To conclude this section ones point out that, in general, the explicit Lagrangian construction is an art rather then a formal technical procedure. Here we study only the Lagrangians for the simple enough supersymmetric theories such as Wess-Zumino model, supersymmetric sigma-model and super Yang-Mills model.

4.2 Wess-Zumino model

The Wess-Zumino model describes a dynamics of chiral and antichiral superfields. The most general action depending on these superfields without higher space-time derivatives looks like

$$S[\Phi,\bar{\Phi}] = \int d^8z K(\Phi,\bar{\Phi}) + \int d^6z W(\Phi) + \int d^6\bar{z}\bar{W}(\bar{\Phi}), \quad (4.24)$$

where $K(\Phi,\bar{\Phi})$ is a real function of complex superfields Φ and $\bar{\Phi}$, $W(\Phi)$ is a function of chiral superfield Φ while $\bar{W}(\bar{\Phi})$ is a function conjugated to $W(\Phi)$.

The inquiring readers can ask, where there are the derivatives in this action at all? To answer, we point out that the derivatives are encoded in the definitions of chiral and antichiral superfields, see Eqs. (3.65), (3.67). Therefore, the term in (4.24) with the function $K(\Phi,\bar{\Phi})$ does contain the space-time derivatives and is responsible for the kinetic terms for the component fields. However, do to the expressions (3.65), (3.67), the terms with the functions $W(\Phi)$, $\bar{W}(\bar{\Phi})$ do not contain the derivatives. Indeed, for any chiral superfield $\Phi(z)$ we have

$$\begin{aligned}\int d^6 z\Phi(z) &= \int d^6 z e^{i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x,\theta) = \int d^6 z[\Phi(x,\theta) \\ &\quad +\text{total space-time derivatives}] \\ &= \int d^6 z\Phi(x,\theta), \end{aligned} \tag{4.25}$$

where Eq. (3.65) has been applied. On these grounds, the term $\int d^8 z K(\Phi,\bar{\Phi})$ is usually called the *kinetic term* while $W(\Phi)$ and $\bar{W}(\bar{\Phi})$ are called *chiral* and *antichiral superpotentials*, respectively.

Historically, the Wess-Zumino model was the first supersymmetric field theory. It corresponds to the following particular choice of the functions $K(\Phi,\bar{\Phi})$, $W(\Phi)$, $\bar{W}(\Phi)$ in the action (4.24):

$$\begin{aligned}K(\Phi,\bar{\Phi}) &= \bar{\Phi}\Phi, \\ W(\Phi) &= \frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3, \\ \bar{W}(\bar{\Phi}) &= \frac{m}{2}\bar{\Phi}^2 + \frac{\lambda}{3!}\bar{\Phi}^3, \end{aligned} \tag{4.26}$$

where m is a mass and λ is a coupling constant. The action (4.24) now reads

$$\begin{aligned}S_{WZ} &= \int d^8 z\bar{\Phi}\Phi + \int d^6 z\left(\frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3\right) \\ &\quad + \int d^6\bar{z}\left(\frac{m}{2}\bar{\Phi}^2 + \frac{\lambda}{3!}\bar{\Phi}^3\right). \end{aligned} \tag{4.27}$$

At present, the models of chiral and antichiral superfields with the action like Eq. (4.24) emerge in the low-energy limit of a superstring theory. In this case the the functions $K(\Phi, \bar{\Phi})$, $W(\Phi)$, $\bar{W}(\bar{\Phi})$ have the special , more complicated form than (4.26).

Let us consider now the derivation of equations of motion corresponding to the action (4.24). As follows from Eq. (3.67), the superfield $\bar{\Phi}(z)$ essentially depends only on $(x, \bar{\theta})$ coordinates. Therefore, the variation of action S with respect to $\bar{\Phi}(z)$ is defined as an integral over the antichiral superspace

$$\delta_{\bar{\Phi}} S = \int d^6\bar{z} \frac{\delta S}{\delta\bar{\Phi}(z)} \delta\bar{\Phi}(z). \tag{4.28}$$

For the variation of the action (4.24) we have

$$\delta_{\bar{\Phi}} S = \int d^8 z \frac{\partial K}{\partial\bar{\Phi}} \delta\bar{\Phi}(z) + \int d^6\bar{z} \frac{\partial\bar{W}}{\partial\bar{\Phi}} \delta\bar{\Phi}(z). \tag{4.29}$$

The integral over the full superspace in (4.29) can be transformed to the integral over the antichiral superspace,

$$\int d^8 z \frac{\partial K}{\partial\bar{\Phi}} \delta\bar{\Phi}(z) = \int d^6\bar{z} \left(-\frac{1}{4} D^2 \frac{\partial K}{\partial\bar{\Phi}}\right) \delta\bar{\Phi}(z). \tag{4.30}$$

Note that the spinor derivatives D_α in Eq. (4.30) do not act on the superfield $\delta\bar{\Phi}$ sine it is antichiral, $D_\alpha \delta\bar{\Phi} = 0$. Therefore, the equation (4.28) can be rewritten as follows

$$\delta_{\bar{\Phi}} S = \int d^6\bar{z} \left(-\frac{1}{4} D^2 \frac{\partial K}{\partial\bar{\Phi}} + \frac{\partial\bar{W}}{\partial\bar{\Phi}}\right) \delta\bar{\Phi}(z). \tag{4.31}$$

As a result, comparing (4.31) with the definition (4.28), we conclude

$$\frac{\delta S}{\delta\bar{\Phi}} = -\frac{1}{4} D^2 \frac{\partial K}{\partial\bar{\Phi}} + \frac{\partial\bar{W}}{\partial\bar{\Phi}}. \tag{4.32}$$

Analogously,

$$\frac{\delta S}{\delta \Phi} = -\frac{1}{4}\bar{D}^2 \frac{\partial K}{\partial \Phi} + \frac{\partial W}{\partial \Phi}. \tag{4.33}$$

Note that the expressions (4.32) and (4.33) are chiral and antichiral, respectively,

$$D_\alpha \frac{\delta S}{\delta \bar{\Phi}} = 0, \qquad \bar{D}_{\dot{\alpha}} \frac{\delta S}{\delta \Phi} = 0. \tag{4.34}$$

$>$From the Eqs. (4.32,4.33) we obtain the equations of motion in the model (4.24)

$$-\frac{1}{4}D^2 \frac{\partial K}{\partial \bar{\Phi}} + \frac{\partial \bar{W}}{\partial \bar{\Phi}} = 0, \qquad -\frac{1}{4}\bar{D}^2 \frac{\partial K}{\partial \Phi} + \frac{\partial W}{\partial \Phi} = 0. \tag{4.35}$$

Further, we will consider only the Wess-Zumino model which is given by the conditions (4.26). In such a case the equations of motion (4.35) are written as follows

$$-\frac{1}{4}D^2 \Phi + m\bar{\Phi} + \frac{\lambda}{2}\bar{\Phi}^2 = 0, \qquad -\frac{1}{4}\bar{D}^2 \bar{\Phi} + m\Phi + \frac{\lambda}{2}\Phi^2 = 0. \tag{4.36}$$

The next problem we will study is the component form of the Wess-Zumino model. For this purpose we represent the chiral and antichiral superfields in the form

$$\begin{aligned} \Phi(z) &= e^{i(\theta\sigma^m\bar{\theta})\partial_m}(A + \theta^\alpha \psi_\alpha + \theta^2 F), \\ \bar{\Phi}(z) &= e^{-i(\theta\sigma^m\bar{\theta})\partial_m}(\bar{A} + \bar{\theta}_{\dot{\alpha}}\psi^{\dot{\alpha}} + \bar{\theta}^2 \bar{F}) \end{aligned} \tag{4.37}$$

and substitute these expressions into the action (4.27)

$$\begin{aligned} S_{WZ} = & \int d^8 z \bar{\Phi}(x,\bar{\theta}) e^{2i(\theta\sigma^m\bar{\theta})\partial_m} \Phi(x,\theta) \\ & + \left\{ \int d^6 z \left(\frac{m}{2}\Phi(x,\theta) + \frac{\lambda}{3!}\Phi^3(x,\theta) \right) \right. \\ & \left. + \text{complex conjugate} \right\}. \end{aligned} \tag{4.38}$$

Here we transferred the operator $e^{-i(\theta\sigma^m\bar{\theta})\partial_m}$ from $\bar{\Phi}$ to Φ using the integration by parts. As to the integrals over chiral and antichiral superspaces, the factors $e^{\pm i(\theta\sigma^m\bar{\theta})\partial_m}$ can be discarded here since we omit the total derivatives. The next step is to apply the identity (4.15) and calculate the spinor derivatives under the integrals. These calculations can be simplified if we take into account the following properties of a Berezin integral. The integral $\int d^4\theta V$ singles out only the coefficient at $\theta^2\bar{\theta}^2$ in a superfield V. Analogously, the integral $\int d^2\theta\Phi$ gives the coefficient at θ^2 in the expansion of a chiral superfield Φ. Therefore, to find the component form of $\int d^8z\bar{\Phi}\Phi$ it is necessary to multiply $\bar{\Phi}$ on Φ and extract the only term proportional to $\theta^2\bar{\theta}^2$. To find the component form of $\int d^6z(\frac{m}{2}\Phi^2+\frac{\lambda}{3!}\Phi^3)$ it is sufficient to extract only terms with θ^2 in the expression $\frac{m}{2}\Phi^2+\frac{\lambda}{3!}\Phi^3$. The result of these computations has the form

$$\begin{aligned} S_{WZ} &= \int d^4x\Big\{-\partial^m\bar{A}\partial_m A-\frac{i}{2}\psi^\alpha\sigma^m{}_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}}+\bar{F}F \\ &\quad +F(mA+\frac{\lambda}{2}A^2)+\bar{F}(m\bar{A}+\frac{\lambda}{2}\bar{A}^2) \\ &\quad -\frac{1}{4}(m+\lambda A)\psi^\alpha\psi_\alpha-\frac{1}{4}(m+\lambda\bar{A})\bar{\psi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}\Big\}. \end{aligned} \quad (4.39)$$

In principle, the action (4.39) can be expressed in terms of standard four-component spinors instead of two-component ones. For this purpose we introduce the Dirac γ-matrices

$$\gamma^m=\begin{pmatrix} 0 & \sigma^m \\ \tilde{\sigma}^m & 0 \end{pmatrix}, \quad \gamma_5=-i\gamma_0\gamma_1\gamma_2\gamma_3=\begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \quad (4.40)$$

and four-component Majorana spinors

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix}, \qquad \bar{\Psi} = \Psi^+ \gamma^0 = \frac{1}{\sqrt{2}} \left(\psi^\alpha, \bar{\psi}_{\dot{\alpha}} \right). \quad (4.41)$$

It is a technical exercise to check that the action (4.39) in term of spinors (4.41) can be rewritten as

$$\begin{aligned} S_{WZ} \;=\; & \int d^4x \Big\{ -\partial^m \bar{A} \partial_m A + \bar{F} F + F(mA + \frac{\lambda}{2} A^2) \\ & + \bar{F}(m\bar{A} + \frac{\lambda}{2} \bar{A}^2) - \frac{i}{2} \bar{\Psi} \gamma^m \partial_m \Psi - \frac{1}{4}(m + \lambda A) \bar{\Psi}(1 \\ & + \gamma_5) \Psi . - \frac{1}{4}(m + \lambda \bar{A}) \bar{\Psi}(1 - \gamma_5) \Psi \Big\}. \end{aligned} \quad (4.42)$$

Let us consider now the equations of motion corresponding to the component action (4.39)

$$\begin{aligned} \Box \bar{A} + F(m + \lambda A) - \frac{1}{4} \lambda \psi^\alpha \psi_\alpha &= 0, \\ i\sigma^m{}_{\alpha\dot{\alpha}} \partial_m \bar{\psi}^{\dot{\alpha}} - (m + \lambda A)\psi_\alpha &= 0, \\ \bar{F} + (mA + \frac{\lambda}{2} A^2) &= 0. \end{aligned} \quad (4.43)$$

We see that the component fields F and $\bar{F}$ enter the equations (4.43) without derivatives and, therefore, have no non-trivial dynamics. These fields can be expressed algebraically from the equations of motion

$$F = -(m\bar{A} + \frac{\lambda}{2} \bar{A}^2), \qquad \bar{F} = -(mA + \frac{\lambda}{2} A^2). \quad (4.44)$$

Such fields are called *auxiliary fields*. Substitute the auxiliary fields (4.44) back into the action (4.39) or (4.42)

$$\begin{aligned}\tilde{S}_{WZ} &= \int d^4x\{\bar{A}(\Box + m^2)A + \frac{1}{2}\bar{\Psi}(i\gamma^m\partial_m + m)\Psi \\ &+\frac{1}{2}m\lambda(A+\bar{A})\bar{A}A + \frac{1}{4}\lambda^2(\bar{A}A)^2 + \\ &+\frac{1}{4}\lambda(A+\bar{A})\bar{\Psi}\Psi + \frac{1}{4}\lambda(A-\bar{A})\bar{\Psi}\gamma_5\Psi\}. \quad (4.45)\end{aligned}$$

As a result, we see that the model under consideration is one of complex scalar field A and Majorana spinor field Ψ with special cubic and quartic scalar self-interactions and special Yukawa coupling. Bosoic and fermionic fields have the same mass m.

To conclude this section we will discuss briefly the role of auxiliary fields. As we see, the theory under consideration can be formulated without these fields. The natural question is why we need them? To clarify this question let us write the supersymmetry transformations of component fields (3.74), (3.75)

$$\begin{aligned}\delta A(x) &= -\epsilon^{\alpha}\psi_{\alpha}(x), \\ \delta\psi_{\alpha}(x) &= -2\epsilon_{\alpha}F(x) - 2i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}A(x), \\ \delta F(x) &= -i\bar{\epsilon}_{\dot{\alpha}}(\tilde{\sigma}^m)^{\dot{\alpha}\alpha}\partial_m\psi_{\alpha}, \\ \delta\bar{A}(x) &= -\bar{\epsilon}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(x), \\ \delta\bar{\psi}_{\dot{\alpha}}(x) &= -2\bar{\epsilon}_{\dot{\alpha}}\bar{F}(x) + 2i\epsilon^{\alpha}\partial_{\alpha\dot{\alpha}}\bar{A}(x), \\ \delta\bar{F}(x) &= i\epsilon^{\alpha}(\tilde{\sigma}^m)_{\alpha\dot{\alpha}}\partial_m\bar{\psi}^{\dot{\alpha}}. \quad (4.46)\end{aligned}$$

Substituting the auxiliary fields from equations of motion

(4.44) to the above relations, one gets

$$\begin{aligned}
\delta A &= -\epsilon^\alpha \psi_\alpha, \\
\delta \bar{A} &= -\bar{\epsilon}_{\dot\alpha} \bar{\psi}^{\dot\alpha}, \\
\delta \psi_\alpha &= 2\epsilon_\alpha (m\bar{A} + \frac{\lambda}{2}\bar{A}^2) - 2i\bar{\epsilon}^{\dot\alpha}\partial_{\alpha\dot\alpha} A, \\
\delta \bar{\psi}_{\dot\alpha} &= 2\epsilon_{\dot\alpha} (mA + \frac{\lambda}{2}A^2) + 2i\epsilon^{\alpha}\partial_{\alpha\dot\alpha} \bar{A}.
\end{aligned} \tag{4.47}$$

The action $\tilde{S}_{WZ}$ (4.45) is automatically invariant under the transformations (4.47). However, let us calculate the commutators of the transformations (4.47) with two different parameters ϵ_1, ϵ_2. The result has the form

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]A = a^m \partial_m A, \tag{4.48}$$

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}]\psi_\alpha = a^m \partial_m \psi_\alpha + i a_{\alpha\dot\alpha} \frac{\tilde{S}_{WZ}}{\delta\bar{\psi}_{\dot\alpha}}, \tag{4.49}$$

where $a^m = 2i(\epsilon_1^\alpha \sigma^m{}_{\alpha\dot\alpha}\bar{\epsilon}_2^{\dot\alpha} - \epsilon_2^\alpha \sigma^m{}_{\alpha\dot\alpha}\epsilon_1^{\dot\alpha})$, $a_{\alpha\dot\alpha} = \sigma^m{}_{\alpha\dot\alpha} a_m$. The equation (4.49) shows that the supersymmetry algebra is broken when the auxiliary fields are eliminated. This algebra is closed only on the equations of motion for spinor fields $\tilde{S}_{WZ}/\delta\bar{\psi}_\alpha = 0$. Hence, the supersymmetry algebra in the theory without auxiliary fields is closed only on-shell (on equations of motion). This explains the role of auxiliary fields: they are responsible for off-shell closure of the supersymmetry algebra.

Let us return back to the superfield formulation of the Wess-Zumino theory (4.27). It has two attractive points in comparison with the component formulation (4.39):

i) Using the superfield formulation we work with a single object $\Phi(z)$ instead of dealing with set of component fields A, Ψ_α, F in the component formulation.

ii) The supersymmetry algebra is automatically closed off-shell.

4.3 Supersymmetric sigma-model

Conventional (non-supersymmetric) sigma-model is defined as a scalar field theory with the action

$$S[\varphi] = -\frac{1}{2}\int d^4x g_{ij}(\varphi)\partial_m\varphi^i\partial^m\varphi^j, \tag{4.50}$$

where φ^i is a set of scalar fields, $i = 1, 2, \ldots, n$. These fields are considered as the coordinates on a Riemann manifold with the metric $g_{ij}(\varphi)$. The action (4.50) is invariant under the following reparametrization transformations

$$\begin{aligned} \varphi^i &\rightarrow \varphi'^i = f^i(\varphi), \\ g_{ij}(\varphi) &\rightarrow g'_{ij}(\varphi') = \frac{\partial\varphi^k}{\partial\varphi'^i}\frac{\partial\varphi^l}{\partial\varphi'^j}g_{kl}(\varphi), \end{aligned} \tag{4.51}$$

where $f^i(\varphi)$ are arbitrary smooth functions. We consider here a supersymmetric generalization of this model.

A supersymmetric sigma-model is a dynamical theory of a set a chiral and antichiral superfields $\Phi^i(z)$ $\bar{\Phi}^{\underline{i}}(z)$ numerated by the indices $i, \underline{i} = 1, 2, \ldots, n$ with the action

$$S_\sigma[\Phi, \bar{\Phi}] = \int d^8z K(\Phi, \bar{\Phi}), \tag{4.52}$$

where $K(\Phi, \bar{\Phi})$ is a real function of n complex variables Φ^i and their conjugate $\bar{\Phi}^{\underline{i}}$. The function $K(\Phi, \bar{\Phi})$ is defined up to the following transformations

$$K(\Phi, \bar{\Phi}) \to K(\Phi, \bar{\Phi}) + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}), \tag{4.53}$$

where $\Lambda(\Phi)$ is an arbitrary holomorphic function of n complex variables Φ^i. Originally, the model (4.52) was suggested by B. Zumino. It can be treated as a generalization of the kinetic term in the Wess-Zumino model (4.27) since here we consider n chiral superfields Φ^i instead one Φ and the action (4.52) includes an arbitrary function K instead of the special one $K = \Phi\bar{\Phi}$.

To clarify the relation of the model (4.52) to the sigma-model (4.50), let us find the component form of the action (4.52). For this purpose we write the component form of superfields as

$$\begin{aligned} \Phi^i &= A^i + \theta^\alpha \psi_\alpha{}^i + \theta^2 F^i + ..., \\ \bar{\Phi}^{\underline{i}} &= \bar{A}^{\underline{i}} + \bar{\theta}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}\underline{i}} + \bar{\theta}^2 \bar{F}^{\underline{i}} + ..., \end{aligned} \tag{4.54}$$

where dots mean the terms with spatial derivatives of component fields. The general procedure of getting the component form of a superfield action is discussed in section 4.1. Following this procedure we represent the action (4.52) as

$$S_\sigma = \int d^4x \frac{1}{16} D^2 \bar{D}^2 K(\Phi, \bar{\Phi}) = \int d^4x L. \tag{4.55}$$

To find the function L we have to fulfil the straightforward calculations according (4.15) taking into account that $\bar{D}_{\dot{\alpha}} \Phi^i =$

0, $D_\alpha \bar{\Phi}^{\underline{i}} = 0$. The final result is written as follows:

$$\begin{aligned} L &= -K_{i\underline{j}}(\partial^m \bar{A}^{\underline{j}} \partial_m A^i - \bar{F}^{\underline{j}} F^i + \frac{i}{4}\psi^{i\alpha}\sigma^m{}_{\alpha\dot{\alpha}} \overleftrightarrow{\partial} \bar{\psi}^{\underline{j}\dot{\alpha}}) \\ & -\frac{1}{4}K_{ij\underline{i}}(\bar{F}^{\underline{i}}\psi^{i\alpha}\psi^j{}_\alpha - i\partial_m A^i \psi^{j\alpha}\sigma^m{}_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}\underline{i}}) \\ & -\frac{1}{4}K_{i\underline{i}\underline{j}}(F^i \bar{\psi}^{\underline{i}}{}_{\dot{\alpha}}\bar{\psi}^{\underline{j}\dot{\alpha}} + i\partial_m \bar{A}^{\underline{i}}\psi^{i\alpha}\sigma^m{}_{\alpha\dot{\alpha}}\bar{\psi}^{\dot{\alpha}\underline{j}}) \\ & +\frac{1}{16}K_{ij\underline{i}\underline{j}}(\psi^{i\alpha}\psi^j{}_\alpha \bar{\psi}^{\underline{i}}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}\underline{j}}), \end{aligned} \tag{4.56}$$

where we have used the notation

$$K_{i_1\ldots i_p \underline{i}_1\ldots \underline{i}_q} = \frac{\partial^{p+q}K(A,\bar{A})}{\partial A^{i_1}\ldots\partial A^{i_p}\partial \bar{A}^{\underline{i}_1}\ldots\partial\bar{A}^{\underline{i}_q}}. \tag{4.57}$$

Comparing Eq. (4.56) with (4.50), we see that in purely scalar sector we get a sigma-model of complex scalar fields A^i with the metric

$$g_{i\underline{j}}(A,\bar{A}) = \frac{\partial^2 K(A,\bar{A})}{\partial A^i \partial \bar{A}^{\underline{j}}}. \tag{4.58}$$

The metric of the form (4.58) defines the so called Kähler geometry and the function $K(A,\bar{A})$ is called *Kähler potential.*

The model (4.56) is described by complex scalar fields A^i, spinor fields ψ^i_α and auxiliary fields F^i. It is convenient to rewrite the Lagrangian (4.56) in some another form. Let us introduce the fields

$$\mathcal{F}^i = F^i - \frac{1}{4}\Gamma^i{}_{jk}\psi^{\alpha j}\psi^k_\alpha, \tag{4.59}$$

where $\Gamma^i{}_{jk}$ are the Cristoffel symbols calculated for the metric $g_{i\underline{j}}$ (4.58) where all derivatives are taken with respect to A^i

but not to $\bar{A}^{\underline{i}}$. Then one gets

$$\begin{aligned} L &= -g_{i\underline{j}}(\partial_m \bar{A}^{\underline{j}} - \partial_m A^i - \bar{\mathcal{F}}^{\underline{j}}\mathcal{F}^i + \frac{i}{4}\psi^{i\dot{\alpha}}\sigma^m{}_{\alpha\dot{\alpha}}\overleftrightarrow{\nabla}_m\psi^{\underline{j}\dot{\alpha}}) \\ &\quad + \frac{1}{16}\mathcal{R}_{i\underline{j}k\underline{l}}\psi^{i\alpha}\psi^k_\alpha\bar{\psi}^{\underline{j}}_{\dot{\alpha}}\bar{\psi}^{\underline{l}\dot{\alpha}}, \end{aligned} \tag{4.60}$$

where $\mathcal{R}_{i\underline{j}k\underline{l}}$ is a curvature tensor

$$\mathcal{R}_{i\underline{j}k\underline{l}} = \frac{\partial^2 g_{i\underline{j}}}{\partial A^k \partial \bar{A}^{\underline{l}}} - g^{p\underline{q}}\frac{\partial g_{i\underline{q}}}{\partial A^k}\frac{\partial g_{p\underline{j}}}{\partial \bar{A}^{\underline{l}}} = K_{i\underline{j}k\underline{l}} - g^{p\underline{q}}K_{ik\underline{q}}K_{p\underline{j}\underline{l}}. \tag{4.61}$$

The symbol ∇_m denotes a target-space covariant derivative

$$\begin{aligned} \nabla_m\psi^i_\alpha &= \partial_m\psi^i_\alpha + \Gamma^i{}_{jl}(\partial_m A^j)\psi^l_\alpha, \\ \nabla_m\bar{\psi}^{\underline{i}}_{\dot{\alpha}} &= \partial_m\bar{\psi}^{\underline{i}}_{\dot{\alpha}} + \Gamma^{\underline{i}}{}_{\underline{j}\underline{l}}(\partial_m \bar{A}^{\underline{i}})\bar{\psi}^{\underline{l}}_{\dot{\alpha}}. \end{aligned} \tag{4.62}$$

The equation (4.89) shows that the supersymmetric sigma-model is expressed completely in terms of geometrical objects as well as the conventional sigma-model.

4.4 Supersymmetric Yang-Mills theories

To formulate the supersymmetric Yang-Mills theory we begin with the model of chiral Φ^i and antichiral $\bar{\Phi}_i$ superfields ($\bar{\Phi}_i = (\Phi^i)^*$) with the action

$$S = \int d^8z\bar{\Phi}_i\Phi^i. \tag{4.63}$$

We assume that these fields carry out a representation of some compact Lie group. This means that they transform as follows

$$\Phi^i \to \Phi'^i = (e^{i\Lambda})^i{}_j\Phi^j, \qquad \bar{\Phi}_i \to \bar{\Phi}'_i = \bar{\Phi}_j(e^{-i\Lambda})^j{}_i, \tag{4.64}$$

where $\Lambda = \Lambda^I T^I$, Λ^I are some constant parameters and $(T^I)^i_j$ are the Hermitian generators satisfying the commutation relation

$$[T^I, T^J] = i f^{IJK} T^K \tag{4.65}$$

with f^{IJK} being the structure constants. It is evident that the action (4.63) is invariant under the transformations of the type (4.64).

Now let us apply a standard trick of localizing the parameters Λ^I, i.e., we assume that they are some functions of x. Note that the superfields, Φ and $\bar{\Phi}$ are chiral and antichiral, respectively. To preserve this property we have to modify the transformations (4.64) considering their as local in superspace

$$\Phi'^i = (e^{i\Lambda})^i{}_j \Phi^j, \qquad \bar{\Phi}'_i = \bar{\Phi}_j (e^{-i\bar{\Lambda}})^j{}_i, \tag{4.66}$$

where Λ is a chiral superfield and $\bar{\Lambda}$ is antichiral one,

$$\begin{aligned} \Lambda &= \Lambda^I(z) T^I, \qquad & \bar{D}_{\dot{\alpha}} \Lambda^I(z) &= 0, \\ \bar{\Lambda} &= \bar{\Lambda}^I(z) T^I, \qquad & D_\alpha \bar{\Lambda}^I(z) &= 0. \end{aligned} \tag{4.67}$$

In the matrix form the equations (4.66) are written as

$$\Phi' = e^{i\Lambda}\Phi, \qquad \bar{\Phi}' = \bar{\Phi} e^{-i\bar{\Lambda}}. \tag{4.68}$$

However, now the action (4.63) is not invariant under the transformations (4.68). Indeed,

$$S' = \int d^8 z \bar{\Phi}' \Phi' = \int d^8 z \bar{\Phi} e^{-i\bar{\Lambda}} e^{i\Lambda} \Phi \neq \int d^8 z \bar{\Phi}\Phi. \tag{4.69}$$

To provide the invariance under the transformations (4.66), (4.67), we modify the action (4.63) by introduction of a gauge superfield. It means we consider the new action

$$S = \int d^8 z \bar{\Phi}_i (e^{2V})^i{}_j \Phi^j \equiv \int d^8 z \bar{\Phi} e^{2V} \Phi, \tag{4.70}$$

where $V = V^I(z)T^I$ and $V^I(z)$ is a real scalar superfield. The transformation law of the superfield V is fixed by requirement that action (4.70) is invariant under (4.68)

$$\int d^8 z \bar{\Phi}' e^{2V'} \Phi' = \int d^8 z \bar{\Phi} e^{-i\bar{\Lambda}} e^{2V'} e^{i\Lambda} \Phi' = \int d^8 z \bar{\Phi} e^{2V} \Phi. \tag{4.71}$$

Hence, we conclude

$$e^{-i\bar{\Lambda}} e^{2V'} e^{i\Lambda} = e^{2V}. \tag{4.72}$$

Thus, if the superfield $V(z)$ transforms by the rule (4.72) and the superfields $\Phi(z)$, $\bar{\Phi}(z)$ transform according to Eq. (4.68), the action (4.70) remains invariant. The superfield $V(z)$ is called *gauge superfield* and the transformations (4.66), (4.67), (4.72) are called *supergauge transformations.* Also $V(z)$ is called *Yang-Mills superfield.* The superfields $\Lambda and \bar{\Lambda}$ are called the gauge superfield parameters.

Since we have introduced a new field into the theory, we have to find the action for this field. Let us introduce the following superfields

$$W_\alpha = -\frac{1}{8}\bar{D}^2(e^{-2V} D_\alpha e^{2V}), \qquad \bar{W}_{\dot{\alpha}} = -\frac{1}{8} D^2 (e^{2V} \bar{D}_{\dot{\alpha}} e^{-2V}). \tag{4.73}$$

It is clear that W_α and $\bar{W}_{\dot\alpha}$ are chiral and antichiral superfields, respectively,

$$\bar{D}_{\dot\alpha} W_\alpha = 0, \qquad D_\alpha \bar{W}_{\dot\alpha} = 0. \tag{4.74}$$

One can prove that they are the Lie-algebra-valued superfields

$$W_\alpha = W^I_\alpha T^I, \qquad \bar{W}_{\dot\alpha} = \bar{W}^I_{\dot\alpha} T^I. \tag{4.75}$$

It is easy to see that W_α, $\bar{W}_{\dot\alpha}$ transform covariantly under the supergauge transformations (4.72). Indeed,

$$\begin{aligned} W'_\alpha &= -\frac{1}{8}\bar{D}^2(e^{-2V'} D_\alpha e^{2V'}) \\ &= e^{i\Lambda} W_\alpha e^{-i\Lambda} - \frac{1}{8} e^{i\Lambda} \bar{D}^2 D_\alpha e^{-i\Lambda} = e^{i\Lambda} W_\alpha e^{-i\Lambda}, \end{aligned} \tag{4.76}$$

where we have used the relation (3.59e) and the chirality of $e^{-i\Lambda}$. Analogously,

$$\bar{W}'_{\dot\alpha} = e^{i\bar\Lambda} \bar{W}_{\dot\alpha} e^{-i\bar\Lambda}. \tag{4.77}$$

As a consequence of Eqs. (4.76), (4.77), the following quantities

$$\mathrm{tr}(W^\alpha W_\alpha), \qquad \mathrm{tr}(\bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}) \tag{4.78}$$

are invariant under the supergauge transformations. Here "tr" denotes the trace over the matrix indices of the generators. Therefore, we can define the action for the gauge superfield in the form

$$S_{SYM} = \frac{1}{4g^2}\int d^6 z\, \mathrm{tr}(W^\alpha W_\alpha) + \frac{1}{4g^2}\int d^6 \bar{z}\, \mathrm{tr}(\bar{W}_{\dot\alpha} \bar{W}^{\dot\alpha}), \tag{4.79}$$

where g is a coupling constant. The superfields (4.76), (4.77) are called the *superfield strengths*. The action (4.79) is considered as action of supersymmetric Yang-Mills theory. Moreover, one can show that (up to total divergence)

$$\int d^6z\,\mathrm{tr}(W^\alpha W_\alpha) = \int d^6\bar{z}\,\mathrm{tr}(\bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}). \tag{4.80}$$

Therefore, the final form of the action of super Yang-Mills theory is

$$S_{SYM}[V] = \frac{1}{2g^2}\int d^6z\,\mathrm{tr}(W^\alpha W_\alpha). \tag{4.81}$$

As a result, the action of super Yang-Mills theory coupled to supersymmetric matter is written as

$$S[V,\Phi,\bar\Phi] = \frac{1}{2g^2}\int d^6z\,\mathrm{tr}(W^\alpha W_\alpha) + \int d^8z\bar\Phi e^{2V}\Phi. \tag{4.82}$$

Further we will discuss two aspects. First, we will find the superfield equations of motion for the theory (4.82) and, second, the component form of the action (4.82).

To derive the superfield equations of motion, we consider the variation of (4.81)

$$\begin{aligned}
\delta S_{SYM}[V] &= \frac{1}{g^2}\int d^6z\,\mathrm{tr}(\delta W^\alpha W_\alpha) \\
&= -\frac{1}{8g^2}\int d^6z\,\bar{D}^2\mathrm{tr}\left(\delta(e^{-2V}D_\alpha e^{2V})W^\alpha\right) \\
&= \frac{1}{2g^2}\int d^8z\,\mathrm{tr}\left(\delta(e^{-2V}D_\alpha e^{2V})W^\alpha\right). \qquad (4.83)
\end{aligned}$$

Here we have used Eq. (4.73) at the second line and equations (4.8), (4.11) at the third one. Due to the identity $e^{-2V}e^{2V}=1$, one gets

$$\delta e^{-2V} = e^{-2V}(\delta e^{2V})e^{-2V}. \qquad (4.84)$$

Applying Eq. (4.84), the variation (4.83) reads

$$\begin{aligned}\delta S_{SYM}[V] &= \frac{1}{2g^2}\int d^8z\,\mathrm{tr}\left\{-e^{-2V}(\delta e^{2V})e^{-2V}D^\alpha e^{2V}\right.\\ &\left.+e^{-2V}(D^\alpha\delta e^{2V})\right\}W_\alpha. \qquad (4.85)\end{aligned}$$

Now, using the property $\mathrm{tr}[e^{-2V}(D^\alpha\delta e^{2V})W_\alpha] = \mathrm{tr}[(D^\alpha e^{2V})W_\alpha e^{-2V}]$ and integration by parts, after some calculations one gets

$$\begin{aligned}\delta S_{SYM}[V] &= -\frac{1}{2g^2}\int d^8z\,\mathrm{tr}(e^{-2V}\delta e^{2V})[D^\alpha W_\alpha \qquad (4.86)\\ &+(e^{-2V}D^\alpha e^{2V})W_\alpha + W_\alpha(e^{-2V}D^\alpha e^{2V})].\end{aligned}$$

>From the variation (4.86) we see that the superfield equation of motion for gauge superfield is given by

$$[D^\alpha W_\alpha + (e^{-2V}D^\alpha e^{2V})W_\alpha + W_\alpha(e^{-2V}D^\alpha e^{2V})] = 0. \qquad (4.87)$$

Let us introduce the operator $\mathcal{D}^\alpha$

$$\mathcal{D}_\alpha = D_\alpha + i\Gamma_\alpha, \qquad (4.88)$$

where

$$i\Gamma_\alpha = e^{-2V}(D_\alpha e^{2V}), \qquad (4.89)$$

which acts on the superfield W_α by the rule

$$\mathcal{D}^\alpha W_\alpha = D^\alpha W_\alpha + i\{\Gamma^\alpha, W_\alpha\}. \qquad (4.90)$$

Using this operator (4.88), the equation of motion of super Yang-Mills theory (4.87) is written in compact form

$$\mathcal{D}^\alpha W_\alpha = 0. \tag{4.91}$$

Now it is a good exercise to show that this equation (4.91) is invariant under supergauge transformations.

$$\mathcal{D}'^\alpha W'_\alpha = e^{i\Lambda}(\mathcal{D}^\alpha W_\alpha)e^{-i\Lambda} = 0. \tag{4.92}$$

Indeed,

$$\begin{aligned}
\mathcal{D}'^\alpha W'_\alpha &= D^\alpha W'_\alpha + e^{-2V'}(D^\alpha e^{2V'})W'_\alpha + W'_\alpha e^{-2V}(D^\alpha e^{2V'}) \\
&= W^\alpha(e^{i\Lambda}W_\alpha e^{-i\Lambda}) \\
&\quad + e^{i\Lambda}e^{-2V}e^{-i\bar{\Lambda}}D^\alpha(e^{i\bar{\Lambda}}e^{2V}e^{-i\Lambda})e^{i\Lambda}W_\alpha e^{-i\Lambda} \\
&\quad + e^{i\Lambda}W_\alpha e^{-i\Lambda}e^{i\Lambda}e^{-2V}e^{-i\bar{\Lambda}}D^\alpha(e^{i\bar{\Lambda}}e^{2V}e^{-i\Lambda}) \\
&= (D^\alpha e^{i\Lambda})W_\alpha e^{-i\Lambda} + e^{i\Lambda}(D^\alpha W_\alpha)e^{-i\Lambda} \\
&\quad - e^{i\Lambda}W_\alpha(D^\alpha e^{-i\Lambda}) \\
&\quad + e^{i\Lambda}e^{-2V}D^\alpha(e^{2V}e^{-i\Lambda})e^{i\Lambda}W_\alpha e^{-i\Lambda} \\
&\quad + e^{i\Lambda}W_\alpha e^{-2V}D^\alpha(e^{2V}e^{-i\Lambda}) \\
&= (D^\alpha e^{i\Lambda})W_\alpha e^{-i\Lambda} + e^{i\Lambda}(D^\alpha W_\alpha)e^{-i\Lambda} \\
&\quad - e^{i\Lambda}W_\alpha(D^\alpha e^{-i\Lambda}) + e^{i\Lambda}e^{-2V}(D^\alpha e^{2V})W_\alpha e^{-i\Lambda} \\
&\quad + e^{i\Lambda}(D^\alpha e^{-i\Lambda})e^{i\Lambda}W_\alpha e^{-i\Lambda} \\
&\quad + e^{i\Lambda}W_\alpha e^{-2V}(D^\alpha e^{2V})e^{-i\Lambda} + e^{i\Lambda}W_\alpha(D^\alpha e^{-i\Lambda}) \\
&= e^{i\Lambda}[D^\alpha W_\alpha + e^{-2V}(D^\alpha e^{2V})W_\alpha \\
&\quad + W_\alpha e^{-2V}(D^\alpha e^{2V})]e^{-i\Lambda} \\
&\quad + (D^\alpha e^{i\Lambda})W_\alpha e^{-i\Lambda} + e^{i\Lambda}(D^\alpha e^{-i\Lambda})e^{-\Lambda}W_\alpha e^{-i\Lambda}.
\end{aligned} \tag{4.93}$$

Due to the identity

$$e^{i\Lambda}(D^\alpha e^{-i\Lambda})e^{i\Lambda}W_\alpha e^{-i\Lambda} = -(D^\alpha e^{i\Lambda})W_\alpha e^{-i\Lambda}, \qquad (4.94)$$

the two terms in the last line of Eq. (4.94) cancel and we arrive at Eq. (4.92). As a result, if $\mathcal{D}^\alpha W_\alpha = 0$ then $\mathcal{D}'^\alpha W'_\alpha = 0$, that means the gauge invariance of the equations of motion (4.91).

The last aspect we discuss in this section is a component form of the action S_{SYM} (4.81). We start with the component expansion of a gauge superfield $V(z)$ which was derived in sect. 3.3

$$\begin{aligned} V(z) \;\; &= \;\; A + \theta^\alpha\psi_\alpha + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha} + \theta^2 F + \bar\theta^2\bar F + (\theta\sigma^m\bar\theta)A_m \\ &\quad + \bar\theta^2\theta^\alpha\lambda_\alpha + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha} + \theta^2\bar\theta^2 D. \end{aligned} \qquad (4.95)$$

All components here take the values in Lie algebra. First of all, we will show that the component form (4.95) can be simplified with the help of gauge transformations. The gauge transformation (4.72) in the infinitesimal form reads

$$e^{V+\delta V} = e^{2V} + i\bar\Lambda e^{2V} - e^{2V}i\Lambda. \qquad (4.96)$$

Expanding all exponentials in (4.96) up to the first order in V, we have

$$\delta V = \frac{i}{2}(\bar\Lambda - \Lambda) + O(V). \qquad (4.97)$$

Since the gauge parameter Λ is a chiral superfield, it can be written as

$$\Lambda = e^{\theta\sigma^m\bar\theta\partial_m}(u(x) + \theta^\alpha\rho_\alpha(x) + \theta^2 f(x)), \qquad (4.98)$$

where $u(x)$, $\rho_\alpha(x)$, $f(x)$ are component fields. Then, the equation (4.97) reads

$$\delta V = \frac{i}{2}(\bar{u} - u) - \frac{i}{2}\theta^\alpha \rho_\alpha + \frac{i}{2}\bar{\theta}_{\dot{\alpha}}\bar{\rho}^{\dot{\alpha}} - \frac{i}{2}\theta^2 f + \frac{i}{2}\bar{\theta}^2 \bar{f}$$
$$+\frac{i}{2}(\theta\sigma^m\bar{\theta})\partial_m(u + \bar{u}) + \ldots, \qquad (4.99)$$

where dots stand for terms with higher space-time derivatives. Comparison of Eq. (4.95) with (4.99) shows that the components A, ψ_α, $\bar{\psi}^{\dot{\alpha}}$, F and $\bar{F}$ in the expansion (4.95) can be done arbitrary and hence they can be gauged away. In the other words, there exists a gauge where these components are equal to zero. As a result, the component form of the superfield $V(z)$ is reduced to

$$V = (\theta\sigma^m\bar{\theta})A_m + \bar{\theta}^2\theta^\alpha\lambda_\alpha + \theta^2\bar{\theta}_{\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} + \theta^2\bar{\theta}^2 D. \qquad (4.100)$$

The gauge where the superfield V has the form (4.100) is called *the Wess-Zumino gauge*. This gauge can also be fixed by equations

$$V| = 0, \quad D_\alpha V| = 0, \quad D^2 V| = 0, \quad \bar{D}^2 V| = 0. \qquad (4.101)$$

It is important to emphasize that in the Wess-Zumino gauge the series for the exponential e^{2V} is reduced to a finite-order polynomial. This allows us to write W_α in the form

$$W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V + \frac{1}{4}\bar{D}^2[V, D_\alpha V]. \qquad (4.102)$$

Next, we apply the general rule

$$\int d^6 \mathcal{L}_c = \int d^4x(-\frac{1}{4}D^2)\mathcal{L}_c, \qquad (4.103)$$

where $\mathcal{L}_c = \frac{1}{2g^2} W^\alpha W_\alpha$ in the case under consideration and W_α is given by Eq. (4.102). To find the component action it is necessary to substitute Eqs. (4.100), (4.102) into (4.103) and compute all the spinor derivatives. The final result is

$$\begin{aligned} S_{SYM} &= \frac{1}{g^2} \int d^4x \, \mathrm{tr} \left\{ -\frac{1}{4} G^{mn} G_{mn} \right. \\ &\left. \quad -i\lambda^\alpha \sigma^m{}_{\alpha\dot{\alpha}} \nabla_m \bar{\lambda}^{\dot{\alpha}} + 2D^2 \right\}, \end{aligned} \tag{4.104}$$

where

$$\begin{aligned} G_{mn} &= \partial_m A_n - \partial_n A_m - i[A_m, A_n], \\ \nabla_m \bar{\lambda}^{\dot{\alpha}} &= \partial_m \bar{\lambda}^{\dot{\alpha}} + i[\bar{\lambda}^{\dot{\alpha}}, A_m]. \end{aligned} \tag{4.105}$$

As a result, we see that the super Yang-Mills theory includes vector field A_m, Majorana spinor $\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix}$ and auxiliary field D. It is worth pointing out that the Wess-Zumino gauge (4.101) does not fix completely a gauge freedom of the theory. The residual gauge symmetry corresponds to gauge freedom in conventional Yang-Mills theory coupled to Majorana spinor.

5 Superfield perturbation theory

5.1 A scheme of perturbation expansion in quantum field theory

The purpose of this section is to remind a scheme of perturbation expansion in quantum field theory. The basic notion of such a construction is the generating functional of Green functions given in terms of path integral.

Let ϕ be a set of fields in the model with action $S[\phi]$. The generating functional of Green functions is defined as the following path integral

$$Z[J] = \int \mathcal{D}\phi e^{i(S[\phi]+\int dx J(x)\phi(x))}. \tag{5.1}$$

The external field $J(x)$ is called a source. The Green functions are expressed on the base of $Z[J]$ by the rule

$$G_n(x_1, \ldots, x_n) = \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta i J(x_1) \ldots \delta i J(x_n)} \bigg|_{J=0}. \tag{5.2}$$

To develop the perturbation expansion of Green functions one writes the action as a sum of quadratic part $S_0[\phi]$, which corresponds to the free action, and interaction $S_{int}[\phi]$ which includes higher powers of fields. The generating functional (5.1) is represented as

$$Z[J] = e^{iS_{int}[\frac{\delta}{\delta iJ}]} Z_0[J], \qquad (5.3)$$

where

$$Z_0[J] = \int \mathcal{D}\phi e^{i(S_0 + \int dx \phi(x) J(x))} = e^{\frac{i}{2}\int dx dx' J(x) D(x,x') J(x')} \qquad (5.4)$$

and $D(x, x')$ in (5.4) is the Feynman propagator.

The perturbation series for Green functions arise when the equations (5.3), (5.4) are substituted into Eq. (5.2)and $e^{iS_{int}[\frac{\delta}{\delta J}]}$ is expanded in power series in $S_{int}[\frac{\delta}{\delta J}]$ and the differentiation with respect to source $J(x)$ is fulfilled. The result is described by Feynman diagrams where the propagator is defined by the quadratic part of the action and the vertices are created by $S_{int}[\phi]$.

There is a simple rule illustrating how to find the Feynman propagator. Consider the following action

$$S_0[\phi] = \frac{1}{2}\int d^4x \phi(x) F_x \phi(x), \qquad (5.5)$$

where F_x is some differential operator acting on space-time coordinate x^m, and introduce the source field $J(x)$ by adding an extra term to (5.5). Then ones get the action

$$S_0[\phi] + \int d^4x J(x)\phi(x). \qquad (5.6)$$

The equation of motion in the model (5.6) is

$$F_x \phi_J(x) + J(x) = 0, \qquad (5.7)$$

where $\phi_J(x)$ is the field in the theory with source $J(x)$. The Feynman propagator is defined by the rule

$$D(x,x') = \frac{\delta\phi_J(x)}{\delta J(x')}. \tag{5.8}$$

Then, Eq. (5.7) leads to

$$F_x D(x,x') = \delta^4(x-x'). \tag{5.9}$$

As a result, we see that the Feynman propagator can be defined as a solution of the equation (5.9).

5.2 Superpropagators in the Wess-Zumino model

Now we turn to the Wess-Zumino model with the action

$$\begin{aligned} S &= \int d^8z\bar{\Phi}\Phi + \int d^6z\left(\frac{m}{2}\Phi^2 + \frac{\lambda}{3!}\Phi^3\right) \\ &\quad + \int d^6\bar{z}\left(\frac{m}{2}\bar{\Phi}^2 + \frac{\lambda}{3!}\bar{\Phi}^3\right) \\ &= S_0[\Phi,\bar{\Phi}] + S_{int}[\Phi,\bar{\Phi}], \end{aligned} \tag{5.10}$$

where

$$\begin{aligned} S_0[\Phi,\bar{\Phi}] &= \int d^8z\bar{\Phi}\Phi + \int d^6z\frac{m}{2}\Phi^2 + \int d^6\bar{z}\frac{m}{2}\bar{\Phi}^2, \\ S_{int}[\Phi,\bar{\Phi}] &= \frac{\lambda}{3!}\int d^6z\Phi^3 + \frac{\lambda}{3!}\int d^6\bar{z}\bar{\Phi}^3. \end{aligned} \tag{5.11}$$

To find the propagators in this theory we will follow the procedure sketched in the previous section. For this purpose we introduce the sources as new external superfields. Since there are two superfields, $\Phi(z)$ and $\bar{\Phi}(z)$, which are chiral and antichiral the corresponding sources $J(z)$ and $\bar{J}(z)$ should also be chiral and antichiral, respectively. The free action in the theory with sources has the form

$$\int d^8z\bar{\Phi}\Phi + \int d^6z\frac{m}{2}\Phi^2 + \int d^6\bar{z}\frac{m}{2}\bar{\Phi}^2 + \int d^6z\Phi J + \int d^6\bar{z}\bar{\Phi}\bar{J}. \tag{5.12}$$

The equations of motion corresponding to the action (5.12) are

$$\begin{aligned} -\frac{1}{4}D^2\Phi_J + m\bar{\Phi}_J + \bar{J} &= 0, \\ -\frac{1}{4}\bar{D}^2\bar{\Phi}_J + m\Phi_J + J &= 0. \end{aligned} \tag{5.13}$$

The theory under consideration is characterized by the matrix propagator with the following elements

$$\begin{aligned} G_{++}(z,z') &= \frac{\delta\Phi_J(z)}{i\delta J(z')}, & G_{+-}(z,z') &= \frac{\delta\Phi_J(z)}{i\delta\bar{J}(z')}, \\ G_{-+}(z,z') &= \frac{\delta\bar{\Phi}_J(z)}{i\delta J(z')}, & G_{--}(z,z') &= \frac{\delta\bar{\Phi}_J(z)}{i\delta\bar{J}(z')}. \end{aligned} \tag{5.14}$$

The indices "+" and "−" mean that the propagator is chiral or antichiral with respect to the corresponding argument. For example,

$$\bar{D}_{\dot{\alpha}(z)}G_{++}(z,z') = 0, \qquad D_{\alpha(z')}G_{--}(z,z') = 0. \tag{5.15}$$

Next, we calculate the variational derivatives of Eqs. (5.13) over the sources $J(z)$ and $\bar{J}(z)$

$$
\begin{aligned}
-\frac{1}{4}D^2G_{++} + mG_{-+} &= 0, \\
-\frac{1}{4}D^2G_{+-} + mG_{--} &= i\delta_-(z,z'), \\
-\frac{1}{4}\bar{D}^2G_{-+} + mG_{++} &= i\delta_+(z,z'), \\
-\frac{1}{4}\bar{D}^2G_{--} + mG_{+-} &= 0. \qquad (5.16)
\end{aligned}
$$

Here δ_+ and δ_- are chiral and antichiral delta-functions defined as follows

$$
\begin{aligned}
\delta_+(z,z') &= -\frac{1}{4}\bar{D}^2\delta^4(x-x')\delta^4(\theta-\theta') \\
&= -\frac{1}{4}\bar{D}^2\delta^8(z-z'), \qquad (5.17) \\
\delta_-(z,z') &= -\frac{1}{4}D^2\delta^4(x-x')\delta^4(\theta-\theta') \\
&= -\frac{1}{4}D^2\delta^8(z-z') \qquad (5.18)
\end{aligned}
$$

Satisfying the (anti)chirality conditions

$$\bar{D}_{\dot\alpha}\delta_+(z,z') = 0, \qquad D_\alpha\delta_-(z,z') = 0. \qquad (5.19)$$

Eqs. (5.16) allow us to express the propagators G_{-+} ,G_{+-} through G_{++}, G_{--} as follows

$$G_{-+} = \frac{1}{4m}D^2G_{++}, \qquad G_{+-} = \frac{1}{4m}\bar{D}^2G_{--}. \qquad (5.20)$$

Substituting Eqs, (5.20) into other equations (5.16), one gets

$$
\begin{aligned}
-\frac{1}{16m}D^2\bar{D}^2G_{--} + mG_{--} = i\delta_-, \\
-\frac{1}{16m}\bar{D}^2D^2G_{++} + mG_{++} = i\delta_+. \qquad (5.21)
\end{aligned}
$$

Next, we apply the identities (3.59) for chiral and antichiral superfields

$$\frac{1}{16}D^2\bar{D}^2G_{--} = \Box G_{--}, \qquad \frac{1}{16}\bar{D}^2D^2G_{++} = \Box G_{++} \quad (5.22)$$

and arrive at the following equations for the propagators

$$\begin{aligned}(\Box - m^2)G_{--} &= -im\delta_-, \\ (\Box - m^2)G_{++} &= -im\delta_+. \end{aligned} \quad (5.23)$$

The solutions of the equations (5.23) are written as

$$G_{--} = \frac{-im}{\Box - m^2 + i\varepsilon}\delta_+, \qquad G_{++} = \frac{-im}{\Box - m^2 + i\varepsilon}\delta_-. \quad (5.24)$$

Here we have used the Feyman prescription defying the causal Green function. Now we have to substitute Eqs. (5.24) into (5.20) and find G_{-+}, G_{+-}. As a result, the matrix propagator in the model under consideration is

$$G = \frac{-i}{\Box - m^2 + i\varepsilon}\begin{pmatrix} m\delta_+ & \frac{1}{4}\bar{D}^2\delta_- \\ \frac{1}{4}D^2\delta_+ & m\delta_- \end{pmatrix}. \quad (5.25)$$

The matrix (5.25) is called the *superpropagator* in the Wess-Zumino model.

The superpropagator (5.21) can be transformed to a more useful form. Note that the superpropagator contains the delta-functions of different chiralities. Therefore we have to consider separately the chiral vertex $\frac{\lambda}{3!}\int d^6z\Phi^3$ and the antichiral one $\frac{\lambda}{3!}\int d^6\bar{z}\bar{\Phi}^3$. However, it is possible to rewrite the superpropagator in such a form where all its matrix elements contain the only delta-function $\delta^8(z - z')$ and, moreover, the superpropagator will have extra D^2, $\bar{D}^2$ factors which help to transform the integrals over chiral or antichiral subspaces to the integrals over the full superspace. For this purpose we substitute the explicit expressions for (anti)chiral delta-functions (5.17), (5.18) into the matrix (5.25)

$$G = \frac{-i}{\Box - m^2 + i\varepsilon} \begin{pmatrix} -\frac{m}{4}\bar{D}^2\delta^8 & -\frac{1}{16}\bar{D}^2 D^2\delta^8 \\ -\frac{1}{16}D^2\bar{D}^2\delta^8 & -\frac{m}{4}D^2\delta^8 \end{pmatrix}. \tag{5.26}$$

Next, we apply the identities (3.59d) in the form

$$\bar{D}^2 = \frac{\bar{D}^2 D^2 \bar{D}^2}{16\Box}, \qquad D^2 = \frac{D^2\bar{D}^2 D^2}{16\Box}. \tag{5.27}$$

As a result, the expression for the superpropagator (5.26) is written in the form

$$G = \frac{-i}{\Box - m^2 + i\varepsilon} \begin{pmatrix} -\frac{m}{4}\frac{\bar{D}^2 D^2 \bar{D}^2}{16\Box}\delta^8 & -\frac{\bar{D}^2 D^2}{16}\delta^8 \\ -\frac{D^2\bar{D}^2}{16}\delta^8 & -\frac{m}{4}\frac{D^2\bar{D}^2 D^2}{16\Box}\delta^8 \end{pmatrix}, \tag{5.28}$$

or

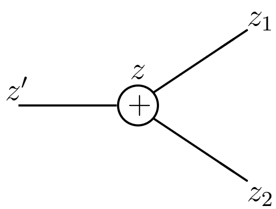
$$G(z,z') = \frac{1}{16} \begin{pmatrix} \frac{m}{4}\frac{\bar{D}^2 D^2 \bar{D}^2}{\Box} & \bar{D}^2 D^2 \\ D^2\bar{D}^2 & \frac{m}{4}\frac{D^2\bar{D}^2 D^2}{\Box} \end{pmatrix} \frac{i\delta^8(z-z')}{\Box - m^2 + i\varepsilon}. \tag{5.29}$$

The expression (5.29) contains a single delta-function $\delta^8(z - z')$ and some number of D- and $\bar{D}$-factors. Although the form (5.29) of the superpropagator looks complicated in comparison with form (5.25) it actually much more convenient for constructing the perturbation theory.

5.3 Supergraphs

The Feynman diagrams in the theory under consideration are constructed on the basis of standard prescriptions. The matrix elements of the superpropagator (5.26) correspond to lines in the diagrams and the integrations $\frac{\lambda}{3!}\int d^6z$, $\frac{\lambda}{3!}\int d^6\bar{z}$ are associated with the corresponding vertices. The only point we have to control is the consistency of chirality and antichirality of propagators with the chirality or antichirality of vertices. It is worth emphasizing that the diagrams are completely formulated in superfield terms. Therefore they are called *supergraphs*.

Let us consider some chiral vertex within a supergraph



(5.30)

There are three internal lines attached to the chiral vertex, therefore the corresponding contribution to the Feynman graph looks like

$$\int d^6z G_{A+}(z', z) G_{+B}(z, z_1) G_{+C}(z, z_2), \qquad (5.31)$$

where $A, B, C = +, -$ are the signs (chiralities) of nearest vertices. The propagators G_{AB} are given by Eq. (5.29). The equation (5.29) shows that each propagator $G_{A+}(z', z)$ has the structure $(...)_{z'}(-\frac{1}{4}\bar{D}_z^2)\delta^8(z'-z)$. Therefore, the factor $\bar{D}_z^2$ can be used under the integral (5.31) to form the full measure d^8z.

Let one of the lines in (5.30) is external one of the whole diagram. In this case we have the contribution

$$\int d^6z\Phi(z)G_{+A}(z,z_1)G_{+B}(z,z_2). \tag{5.32}$$

From Eq. (5.29) we see that any propagator $G_{+A}(z,z_1)$ has the factor $-\frac{1}{4}\bar{D}_z^2(...)$. Due to chirality of the superfield $\Phi(z)$, one can use the factor $-\frac{1}{4}\bar{D}_z^2$ to obtain the integral over the full superspace $\int d^6z(-\frac{1}{4}\bar{D}^2)(...) = \int d^8z(...)$.

Analogous consideration is also valid for antichiral vertex. The factors $-\frac{1}{4}D_z^2$ can be used to restore the full superspace measure d^8z in the correspond vertices. As a result, we see that all vertices in supergraphs correspond to the integrals over full superspace only.

We would like to note also that each matrix element of (5.29) contains the operator D_α and $\bar{D}_{\bar{\alpha}}$. Some of them are used to form the full superspace measure, other will treated as the operators in vertices acting on the lines. It corresponds to the following rules for the lines in supergraphs

$$\Phi\bar{\Phi}\text{-line:}\qquad z \overline{} z' = \frac{i}{\Box - m^2}\delta^8(z - z') = K_{+-},$$

$$\Phi\Phi\text{-line:}\qquad z \overset{\frac{mD^2}{4\Box}}{\overline{+}} z' = K_{++} = \frac{1}{4}D^2\frac{im}{\Box(\Box - m^2)}\delta^8(z - z')$$

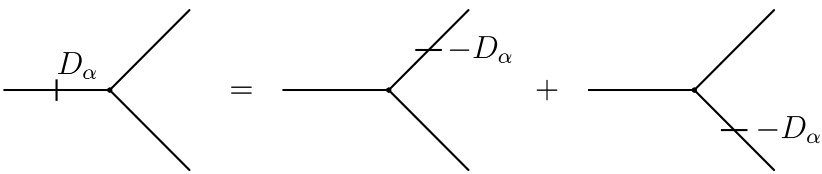
$$= \frac{mD^2}{4\Box}K_{+-}, \tag{5.33}$$

$$\bar{\Phi}\bar{\Phi}\text{-line:}\qquad z \overset{\frac{m\bar{D}^2}{4\Box}}{\overline{+}} z' = K_{--} = \frac{1}{4}\bar{D}^2\frac{im}{\Box(\Box - m^2)}\delta^8(z - z')$$

$$= \frac{m\bar{D}^2}{4\Box}K_{+-}. \tag{5.34}$$

As a result, the lines can bear the factors D_α, $\bar{D}_{\dot{\alpha}}$ acting on the function K_{+-}. The matrix with the elements K_{++}, K_{+-}, K_{-+}, K_{--} is called the improved superpropagator.

The factors D_α, $\bar{D}_{\dot{\alpha}}$ can be transferred from one line to another with the help of integration by parts. For example,



This manipulation allows us to remove all D-factor from at least one Grassmann δ-function and integrate over one of $d^4\theta$. Since each of the propagators contains the delta-function $\delta^4(\theta - \theta')$, we can decrease a number of integrations over $d^4\theta$.

As usual, one can make transformation from coordinate to the momentum space with the help of standard Fourier transform. In the momentum picture all spatial derivatives ∂_m become the operators of multiplication on $-ip_m$

$$\begin{aligned}
\partial_m &\rightarrow -ip_m, \\
D_\alpha &\rightarrow D_\alpha(p) = \partial_\alpha + (\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} p_m, \\
\bar{D}_{\dot{\alpha}} &\rightarrow \bar{D}_{\dot{\alpha}}(p) = -\bar{\partial}_{\dot{\alpha}} - \theta^\alpha \sigma^m{}_{\alpha\dot{\alpha}} p_m.
\end{aligned} \tag{5.35}$$

It is easy to check the following anticommutation relations for the operators (5.35)

$$\begin{gathered}
\{D_\alpha, D_\beta\} = 0, \qquad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \\
\{D_\alpha(p), \bar{D}_{\dot{\alpha}}(p)\} = -2\sigma^m{}_{\alpha\dot{\alpha}} p_m.
\end{gathered} \tag{5.36}$$

All further considerations are analogous to the ones in the conventional quantum field theory and we do not discuss that. Here we emphasized only these aspects which are specific just to superfield theories.

5.4 Non-renormalization theorem

A non-renormalization theorem in supersymmetric quantum field theory concerns superfiled structure of the effective action and, as a consequence, explains the specific features of quantum corrections in component form. In particular, this theorem explains the cancelations of divergences in such theories.

By definition, the effective action is a generating functional of connected one particle irreducible amputated Green functions. In conventional quantum field theory it has the following general form

$$\Gamma[\phi] = \sum_{n=2}^{\infty} \frac{1}{n!} \int dx_1 \dots dx_n \Gamma^{(n)}(x_1, \dots, x_n) \phi(x_1) \dots \phi(x_n), \tag{5.37}$$

where $\Gamma^{(n)}(x_1, \dots, x_n)$ are connected one-particle irreducible amputated Green functions and $\phi(x)$ denotes all fields entering the theory.

The non-renormalization theorem in $\mathcal{N} = 1$ supersymmetric quantum field theory states that each term in the effective action can be expressed as an integral over a single $d^4\theta$. In the other words, any supergraph contributing to effective action can be presented in the form with a single integral over full superspace. It means that the superfield effective action is always local in anticommuting coordinates.

The general idea of the proof is based on the observation that the superpropagator (5.29) contains the Grassmann delta-function $\delta^4(\theta - \theta')$ allowing to fulfill all integrals over anticommuting variables except one[1].

Proof. Consider arbitrary one-particle irreducible L-loop supergraph. As we know, at each vertex there is the inte-

[1]We consider the non-renormalization theorem only for Wess-Zumino model. General consideration is given e.g. in the books S.J. Gates, M.T. Grisaru, M. Roćek, W. Siegel, *Superspace or One Thousand and One Lessons in Supersymmetry*, Addison-Wisley, 1983; hep-th/0108200 and I.L. Buchbinder, S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, Bristol, IOP Publ., 1998.

gration over $d^4\theta$ and each line includes $\delta^4(\theta - \theta')$ with some number of factors $D_\alpha, \bar{D}_{\dot{\alpha}}$ acting on this delta-function. We omit here the dependence of a supergraph on the internal and external momenta since it has no relation to the problem under discussion. Consider in the supergraph a fixed loop involving, say, n vertices. It is clear that there will be a cycle of Grassmann delta-functions associated with propagators

$$\delta^4(\theta_1 - \theta_2)\delta^4(\theta_2 - \theta_3)\dots\delta^4(\theta_n - \theta_1) \tag{5.38}$$

with some number of derivatives D_α, $\bar{D}_{\dot{\alpha}}$ acting on δ-functions. Integrating by parts, we can transfer all the spinor derivatives acting on $\delta^4(\theta_1 - \theta_2)$ to the delta functions $\delta^4(\theta_2 - \theta_3)$ or $\delta^4(\theta_n - \theta_1)$ or to the external lines of the loop supergraph. Then, one integrates over θ_2 and uses $\delta^4(\theta_1 - \theta_2)$ to replace θ_2 by θ_1 everywhere. After that, we continue this process $n-3$ times with the remaining delta-functions. As a result, the cycle is reduced to a single delta-function. The expression for the supergraph takes the following schematic form

$$\int d^4\theta_1 \prod_A \int d^4\theta_A f(\theta_1, \theta_A)[D\dots D\bar{D}\dots\bar{D}\delta^4(\theta_n - \theta_1)]|_{\theta_n=\theta_1}. \tag{5.39}$$

Here the index A enumerates the vertices external to the given loop which appear with some factors $f(\theta_1, \theta_A)$. The expression $[D\dots D\bar{D}\dots\bar{D}\delta^4(\theta_n - \theta_1)]|_{\theta_n=\theta_1}$ in (5.39) can be easily evaluated. First, using the anticommutation relations among the spinor derivatives D_α, $\bar{D}_{\dot{\alpha}}$, (3.55), any product of them is reduced to an expression involving no more than four such factors. Since $\delta^4(\theta_n - \theta_1) = (\theta_n - \theta_1)^2(\bar{\theta}_n - \bar{\theta}_1)^2$, we need exactly two derivatives D_α and two $\bar{D}_{\dot{\alpha}}$ ones, otherwise the expression $[D \ldots D\bar{D} \ldots \bar{D}\delta^4(\theta_n - \theta_1)]|_{\theta_n=\theta_1}$ is zero. Then we use the identity

$$\frac{1}{16} D^2 \bar{D}^2 \delta^4(\theta_n - \theta_1)|_{\theta_n=\theta_1} = 1. \tag{5.40}$$

As a result, the loop is shrunk into a point with respect to the anticommuting variables.

Continuing the above procedure loop-by-loop one reduces the whole supergraph to a point in θ-space and the total contribution takes the form

$$\int d^4p_1 \ldots d^4p_L \int d^4\theta \mathcal{F}(p_1, \ldots, p_L, \theta, \bar{\theta}), \tag{5.41}$$

where p_i are the internal momenta associated with the loops. This finalizes the proof of the theorem.

Let us discuss the consequences of this theorem.

1. According to the non-renormalization theorem, the effective action is represented in the form with single integration over $d^4\theta$. For example, in the Wess-Zumino model it has the following general structure

$$\begin{aligned}\Gamma[\Phi, \bar{\Phi}] \;=\; & \sum_{n=2}^{\infty} \int d^4x_1 \ldots d^4x_n \int d^4\theta \mathcal{G}_n(x_1, \ldots, x_n) F_1(x_1, \theta) \\ & \ldots F_n(x_n, \theta), \end{aligned} \tag{5.42}$$

where $\mathcal{G}_n$ are the translationally invariant functions of the Minkowski space coordinates and F_1, $F_2, \ldots$, F_n are the local functions of superfields Φ, $\bar{\Phi}$ and their covariant derivatives

$$\begin{aligned} F_i &= F_i(\Phi_i, \bar{\Phi}_i, D_M\Phi_i, D_M\bar{\Phi}_i, \ldots), \\ \Phi_i &= \Phi_i(x_i, \theta, \bar{\theta}), \quad \bar{\Phi}_i = \bar{\Phi}_i(x_i, \theta, \bar{\theta}). \end{aligned} \tag{5.43}$$

2. All vacuum supergraphs vanish. Indeed, according to the non-renormalization theorem, any vacuum supergraph is written as

$$A \int d^4\theta \cdot 1, \tag{5.44}$$

where A is a θ-independent loop momentum integral. But $\int d^4\theta \, 1 \equiv 0$.

3. There are no (anti)chiral divergences in the Wess-Zumino model. Indeed, the divergences are (quasi)local in space-time, therefore in given case any (anti)chiral divergent supergraph contribution to effective action must be local in Minkowski space and hence is written as

$$\int d^4x d^2\theta \mathcal{W}(\Phi) + \int d^4x d^2\bar{\theta} \bar{\mathcal{W}}(\bar{\Phi}). \tag{5.45}$$

But all expressions of the form (5.45) are forbidden by the non-renormalization theorem, since the general structure of the effective action is given by (5.42) with the integration over $d^4\theta$, not over $d^2\theta$ or $d^2\bar{\theta}$.

At first sight, the renormalization theorem prohibits any contributions to the (anti)chiral potentials. In fact, it is not true and the finite corrections are possible. For example, let us consider the identity

$$-\frac{1}{4}\int d^8z \frac{D^2}{\Box} G = \int d^6 z G, \tag{5.46}$$

where G is a chiral superfield. Therefore, a finite term of the type $\int d^8z(-\frac{1}{4}\frac{D^2}{\Box})$ is unprohibited in (5.42) and can give a contribution to the purely chiral quantum corrections. We emphasize that this finite term is non-local in x-space while the divergent terms are always local[2].

4. Non-renormalization theorem immediately shows that there is the only renormalization constant in the Wess-Zumino model.

The standard arguments tell us that the renormalization of the Wess-Zumino model is described by the transformation

$$\Phi = z_1^{1/2}\Phi_R, \quad \bar{\Phi} = z_1^{1/2}\bar{\Phi}_R, \quad m = z_m m_R, \quad \lambda = z_\lambda \lambda_R, \tag{5.47}$$

where the label R means the renormalized quantity and z_1, z_m, z_λ are renormalization constants. Substituting Eqs. (5.47) into the Wess-Zumino action (4.27), we obtain the corresponding renormalized action

$$\begin{aligned} S_R &= \int d^8z\, z_1\bar{\Phi}_R\Phi_R + \Big[\int d^6z(\frac{1}{2}z_1 z_m m_R\Phi_R^2 \\ &\quad + \frac{1}{3!}z_\lambda z_1^{3/2}\lambda_R\Phi_R^3) + \text{complex conjugate}\Big], \end{aligned} \tag{5.48}$$

But, according to the non-renormalization theorem, there are no divergent contributions to the (anti)chiral potentials, i.e.,

$$\frac{1}{2}m\Phi^2 + \frac{\lambda}{3!}\Phi^3 = \frac{1}{2}m_R\Phi_R^2 + \frac{\lambda_R}{3!}\Phi_R^3. \tag{5.49}$$

---

[2]The calculations of chiral quantum corrections to effective action is considered in the book I.L. Buchbinder, S.M. Kuzenko, *Ideas and Methods of Supersymmetry and Supergravity or a Walk Through Superspace*, Bristol, IOP Publ., 1998.

Hence,

$$z_1 z_m = 1, \qquad z_1^{3/2} z_\lambda = 1, \tag{5.50}$$

and we conclude that the model is characterized by the only independent renormalization constant z_1.

6 Problems

6.1 Lorentz and Poincare groups

Problem 1.1. Let the matrices Λ_1 and Λ_2 satisfy the relation

$$\Lambda^{\mathrm{T}}\eta\Lambda = \eta, \tag{P1}$$

where $\eta = \mathrm{diag}(-1, 1, 1, 1)$. Show that the matrix $\Lambda_2\Lambda_1$ satisfy the relation (P1) as well.

Problem 1.2. Let (a, Λ) be some non-homogeneous Lorentz transformation. Show that the set of such transformations form a group with the following multiplication low

$$(a_2, \Lambda_2)(a_1, \Lambda_1) = (\Lambda_2 a_1 + a_2, \Lambda_2\Lambda_1). \tag{P2}$$

Problem 1.3. Prove the identities

$$N\varepsilon N^{\mathrm{T}} = \varepsilon, \qquad N^{\mathrm{T}}\varepsilon^{-1}N = \varepsilon^{-1}, \tag{P3}$$

where $\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and N is a complex 2×2 matrix with the unit determinant, $\det N = 1$.

Problem 1.4. Prove the equality

$$\varphi_1'^{\,\alpha}\varphi_{2\alpha}' = \varphi_1^{\alpha}\varphi_{2\alpha}, \tag{P4}$$

where $\varphi_\alpha' = N_\alpha{}^\beta\varphi_\beta$ and $\varphi^\alpha = \varepsilon^{\alpha\beta}\varphi_\beta$.

Problem 1.5. Prove the relations

$$\varphi_1^{\alpha}\varphi_{2\alpha} = -\varphi_2^{\alpha}\varphi_{1\alpha}, \qquad \chi_{1\dot\alpha}\chi_2^{\dot\alpha} = -\chi_{2\dot\alpha}\chi_1^{\dot\alpha}. \tag{P5}$$

Problem 1.6. Let $(\sigma_m)_{\alpha\dot\alpha} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ are the following matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Introduce also the matrices $\tilde\sigma_m$ by the rules

$$(\tilde\sigma_m)^{\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}(\sigma_m)_{\beta\dot\beta}, \quad \tilde\sigma_m = (\sigma_0, -\sigma_1, -\sigma_2, -\sigma_3). \tag{P6}$$

Prove the identities

$$\begin{aligned}
(\sigma_m\tilde\sigma_n + \sigma_n\tilde\sigma_m)_\alpha{}^\beta &= -2\eta_{mn}\delta_\alpha^\beta, \\
(\tilde\sigma_m\sigma_n + \tilde\sigma_n\sigma_m)^{\dot\alpha}{}_{\dot\beta} &= -2\eta_{mn}\delta_{\dot\beta}^{\dot\alpha}, \\
\mathrm{tr}(\sigma_m\tilde\sigma_n) &= -2\eta_{mn}, \\
(\sigma^m)_{\alpha\dot\alpha}(\tilde\sigma_m)^{\dot\beta\beta} &= -2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}.
\end{aligned} \tag{P7}$$

Problem 1.7. Any vector index m can be transformed to a couple of spinor ones $\alpha\dot{\alpha}$ by the rules

$$
\begin{aligned}
V_{\alpha\dot{\alpha}} &= (\sigma^m)_{\alpha\dot{\alpha}} V_m, \qquad & V_m &= -\frac{1}{2}(\tilde{\sigma}_m)^{\dot{\alpha}\alpha} V_{\alpha\dot{\alpha}} \\
W_{\alpha\dot{\alpha}} &= (\sigma^m)_{\alpha\dot{\alpha}} W_m, \qquad & W_m &= -\frac{1}{2}(\tilde{\sigma}_m)^{\dot{\alpha}\alpha} W_{\alpha\dot{\alpha}}.
\end{aligned} \tag{P8}
$$

Check the identity

$$
V_m W^m = -\frac{1}{2} V_{\alpha\dot{\alpha}} W^{\alpha\dot{\alpha}}. \tag{P9}
$$

Problem 1.8. Consider the matrices

$$
\begin{aligned}
(\sigma_{mn})_\alpha{}^\beta &= -\frac{1}{4}(\sigma_m\tilde{\sigma}_n - \sigma_n\tilde{\sigma}_m)_\alpha{}^\beta, \\
(\tilde{\sigma}_{mn})^{\dot{\alpha}}{}_{\dot{\beta}} &= -\frac{1}{4}(\tilde{\sigma}_m\sigma_n - \tilde{\sigma}_n\sigma_m)^{\dot{\alpha}}{}_{\dot{\beta}},
\end{aligned} \tag{P10}
$$

where the matrices σ_m are given by (P6). Show that any antisymmetric tensor $H_{mn} = -H_{nm}$ is equivalent to a pair of two symmetric spin-tensors $h_{\alpha\beta}$, $h_{\dot{\alpha}\dot{\beta}}$ and they are related as follows

$$
\begin{gathered}
H_{mn} = (\sigma_{mn})_{\alpha\beta} h^{\alpha\beta} - (\tilde{\sigma}_{mn})_{\dot{\alpha}\dot{\beta}} h^{\dot{\alpha}\dot{\beta}}, \\
h_{\alpha\beta} = \tfrac{1}{2}(\sigma^{mn})_{\alpha\beta} H_{mn}, \qquad h_{\dot{\alpha}\dot{\beta}} = -\tfrac{1}{2}(\tilde{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}} H_{mn}.
\end{gathered} \tag{P11}
$$

Problem 1.9. Let $F_{mn} = \partial_m A_n - \partial_n A_m$ be an electromagnetic strength tensor. Consider the corresponding spin-tensor components of the strength tensor

$$
F_{\alpha\beta} = \frac{1}{2}(\sigma^{mn})_{\alpha\beta} F_{mn}, \qquad F_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2}(\tilde{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}} F_{mn}. \tag{P12}
$$

a. Express the spin-tensors $F_{\alpha\beta}$, $F_{\dot\alpha\dot\beta}$ through the following objects

$$A_{\alpha\dot\alpha} = (\sigma^m)_{\alpha\dot\alpha} A_m, \qquad \partial_{\alpha\dot\alpha} = (\sigma^m)_{\alpha\dot\alpha}\partial_m. \tag{P13}$$

Answer:

$$\begin{aligned} F_{\alpha\beta} &= -\frac{1}{4}(\partial_{\alpha\dot\alpha} A^{\dot\alpha}_{\beta} + \partial_{\beta\dot\alpha} A^{\dot\alpha}_{\alpha}), \\ F_{\dot\alpha\dot\beta} &= \frac{1}{4}(\partial^{\alpha}_{\dot\alpha} A_{\alpha\dot\beta} + \partial^{\alpha}_{\dot\beta} A_{\alpha\dot\alpha}). \end{aligned} \tag{P14}$$

b. Express $F^{mn}F_{mn}$ through the spin-tensors $F_{\alpha\beta}$, $F_{\dot\alpha\dot\beta}$. Answer:

$$F^{mn}F_{mn} = 2F^{\alpha\beta}F_{\alpha\beta} + 2F^{\dot\alpha\dot\beta}F_{\dot\alpha\dot\beta}. \tag{P15}$$

Problem 1.10. Show that the expression $u^m = \varphi^\alpha(\sigma^m)_{\alpha\dot\alpha}\chi^{\dot\alpha}$ is a four-vector with respect to the Lorentz rotations.

Problem 1.11. Prove that under the non-homogeneous Lorentz transformations a vector field $t^m(x)$ transforms by the rule

$$\delta t^m(x) = -a^n\partial_n t^m(x) + \omega^m{}_n t^n(x) - \omega^n{}_k x^k \partial_n t^m(x). \tag{P16}$$

Check that the transformations (P16) can be written as

$$\delta t^m(x) = -ia^r(P_r)^m{}_n t^n(x) + \frac{i}{2}\omega^{rs}(J_{rs})^m{}_n t^n(x), \tag{P17}$$

where the operators P_r, J_{rs} are given by

$$\begin{aligned}
(P_r)^m{}_n &= \delta^m_n(-i\partial_r), \\
(J_{rs})^m{}_n &= \eta_{rk}x^k(P_s)^m{}_n - \eta_{sk}x^k(P_r)^m{}_n + (M_{rs})^m{}_n, \\
(M_{rs})^m{}_n &= i(\delta^m_s\eta_{rn} - \delta^m_r\eta_{sn}).
\end{aligned} \tag{P18}$$

Problem 1.12. Show that the operators P_r, J_{rs} given by Eqs. (P18) satisfy the following commutation relations

$$\begin{aligned}
[P_r, P_s] &= 0, \\
[J_{rs}, P_m] &= i(\eta_{rm}P_s - \eta_{sm}P_r), \\
[J_{mn}, J_{rs}] &= i(\eta_{mr}J_{ns} - \eta_{ms}J_{nr} + \eta_{ns}J_{mr} - \eta_{nr}J_{ms}).
\end{aligned} \tag{P19}$$

Problem 1.13. Consider the operators [1]

$$C_1 = P^mP_m \equiv P^2, \qquad C_2 = W^mW_m \equiv W^2, \tag{P20}$$

where $W^m = \frac{1}{2}\varepsilon^{mnrs}P_nJ_{rs}$. Prove the following relations

$$[P_m, W_n] = 0, \tag{P21a}$$

$$[J_{mn}, W_r] = i(\eta_{nr}W_m - \eta_{mr}W_n), \tag{P21b}$$

$$[W_m, W_n] = i\varepsilon_{mnrs}W^rP^s, \tag{P21c}$$

$$[C_1, P_m] = 0, \qquad [C_1, J_{mn}] = 0, \tag{P21d}$$

$$[C_2, P_m] = 0, \qquad [C_2, J_{mn}] = 0. \tag{P21e}$$

[1] The operators C_1 and C_2 given by Eq. (P20) are called Casimir operators of the Poincare group.

6.2 Superspace and superfields

Problem 2.1. Prove the relations

$$\begin{aligned} \theta_\alpha\theta_\beta &= \frac{1}{2}\varepsilon_{\alpha\beta}\theta^2, & \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} &= -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}\bar\theta^2, \\ \theta^\alpha\theta^\beta &= -\frac{1}{2}\varepsilon^{\alpha\beta}\theta^2, & \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} &= \frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}\bar\theta^2, \end{aligned} \tag{P22}$$

where $\theta^2 \equiv \theta^\alpha\theta_\alpha = \varepsilon^{\alpha\beta}\theta_\beta\theta_\alpha$, $\bar\theta^2 \equiv \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}\bar\theta^{\dot\alpha}$.

Problem 2.2. Let the supercharges are given by

$$Q_\alpha = i\partial_\alpha + \sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m, \qquad \bar Q_{\dot\alpha} = -i\bar\partial_{\dot\alpha} - \theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m. \tag{P23}$$

Let the covariant spinor derivatives are searched in the form

$$D_\alpha = c_1\partial_\alpha + c_2\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m, \qquad \bar D_{\dot\alpha} = c_3\bar\partial_{\dot\alpha} + c_4\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m \tag{P24}$$

with some unknown coefficients c_1, c_2, c_3, c_4. Find these coefficients from the following anticommutation relations of spinor derivatives (P24) with the supercharges (P23)

$$\begin{aligned} \{D_\alpha, Q_\beta\} &= 0, & \{D_\alpha, \bar Q_{\dot\beta}\} &= 0, \\ \{\bar D_{\dot\alpha}, Q_\beta\} &= 0, & \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\} &= 0. \end{aligned} \tag{P25}$$

Problem 2.3. Show that the covariant spinor derivatives

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m = \partial_\alpha + i\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}, \\ \bar D_{\dot\alpha} &= -\bar\partial_{\dot\alpha} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m = -\bar\partial_{\dot\alpha} - i\theta^\alpha\partial_{\alpha\dot\alpha} \end{aligned} \tag{P26}$$

satisfy the following (anti)commutation relations

$$
\begin{aligned}
\{D_\alpha, D_\beta\} &= 0, \qquad \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0, \\
[D_\alpha, \partial_m] &= 0, \qquad [\bar{D}_{\dot{\alpha}}, \partial_m] = 0, \\
\{D_\alpha, \bar{D}_{\dot{\alpha}}\} &= -2i\partial_{\alpha\dot{\alpha}} = 2P_{\alpha\dot{\alpha}}.
\end{aligned} \tag{P27}
$$

Problem 2.4. Prove the following identities with the covariant derivatives

$$
\begin{gathered}
D^2\bar{D}_{\dot{\alpha}}D^2 = 0, \qquad \bar{D}^2 D_\alpha \bar{D}^2 = 0, \\
D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot{\alpha}} D^2 \bar{D}^{\dot{\alpha}}, \\
D^2\bar{D}^2 + \bar{D}^2 D^2 - 2D^\alpha \bar{D}^2 D_\alpha = 16\Box, \\
D^2\bar{D}^2 D^2 = 16 D^2\Box, \qquad \bar{D}^2 D^2 \bar{D}^2 = 16\bar{D}^2\Box, \\
[D^2, \bar{D}_{\dot{\alpha}}] = -4i\partial_{\alpha\dot{\alpha}} D^\alpha, \qquad [\bar{D}^2, D_\alpha] = 4i\partial_{\alpha\dot{\alpha}}\bar{D}^{\dot{\alpha}}.
\end{gathered} \tag{P28}
$$

Problem 2.5. Show that the operators

$$
\begin{aligned}
\mathcal{P}_{(+)} &= \frac{1}{16\Box}\bar{D}^2 D^2, \quad \mathcal{P}_{(-)} = \frac{1}{16\Box} D^2\bar{D}^2, \\
\mathcal{P}_{(0)} &= -\frac{1}{8\Box} D^\alpha \bar{D}^2 D_\alpha
\end{aligned} \tag{P29}
$$

satisfy the following conditions for projection operators

$$
\mathcal{P}_{(+)} + \mathcal{P}_{(-)} + \mathcal{P}_{(0)} = 1, \quad \mathcal{P}_{(i)}\mathcal{P}_{(j)} = \delta_{ij}\mathcal{P}_{(i)}, \quad i, j = +, -, 0. \tag{P30}
$$

Problem 2.6. Consider the operators

$$\begin{aligned}
P_m &= -i\partial_m, \\
J_{mn} &= i(x_n\partial_m - x_m\partial_n + (\sigma_{mn})^{\alpha\beta}\theta_\alpha\partial_\beta - (\tilde{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}}\bar{\theta}_{\dot{\alpha}}\bar{\partial}_{\dot{\beta}}), \\
Q_\alpha &= i\partial_\alpha + (\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m, \\
\bar{Q}_{\dot{\alpha}} &= -i\bar{\partial}_{\dot{\alpha}} - \theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m.
\end{aligned} \tag{P31}$$

Show that the operators (P31) obey the (anti)commutation relations of Poincare superalgebra

$$\begin{aligned}
[P_m, P_n] &= 0, \qquad [J_{mn}, P_r] = i\eta_{mr}P_n - i\eta_{nr}P_m, \\
[J_{mn}, J_{rs}] &= i\eta_{mr}J_{ns} - i\eta_{ms}J_{nr} + i\eta_{ns}J_{mr} - i\eta_{nr}J_{ms}, \\
[J_{mn}, Q_\alpha] &= i(\sigma_{mn})_\alpha{}^\beta Q_\beta, \quad [P_m, Q_\alpha] = 0, \\
[J_{mn}, \bar{Q}^{\dot{\alpha}}] &= i(\tilde{\sigma}_{mn})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{Q}^{\dot{\beta}}, \quad [P_m, \bar{Q}^{\dot{\alpha}}] = 0, \\
\{Q_\alpha, Q_\beta\} &= 0, \quad \{\bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}}\} = 0, \\
\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2(\sigma^m)_{\alpha\dot{\alpha}}P_m.
\end{aligned} \tag{P32}$$

Problem 2.7. Consider a superfield

$$\Phi(y,\theta) = A(y) + \theta^\alpha\psi_\alpha(y) + \theta^2F(y), \tag{P33}$$

where $y^m = x^m + i\theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = x^m + i(\theta\sigma^m\bar{\theta})$. Prove the identity

$$\begin{aligned}
\Phi(x,\theta,\bar{\theta}) \equiv\ & \Phi(x + i(\theta\sigma\bar{\theta}), \theta) = A(x) + \theta^\alpha\psi_\alpha(x) + \theta^2F(x) \\
& + i(\theta\sigma^m\bar{\theta})\partial_m A(x) + \frac{i}{2}\theta^2\bar{\theta}_{\dot{\alpha}}(\tilde{\sigma}^m)^{\dot{\alpha}\alpha}\partial_m\psi_\alpha(x) \\
& - \frac{1}{4}\theta^2\bar{\theta}^2\Box A(x).
\end{aligned} \tag{P34}$$

Problem 2.8. Prove the identities

$$\begin{aligned}
e^{-i(\theta\sigma^m\bar{\theta})\partial_m} D_\alpha e^{i(\theta\sigma^m\bar{\theta})\partial_m} &= \partial_\alpha + 2i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\
e^{-i(\theta\sigma^m\bar{\theta})\partial_m} \bar{D}_{\dot{\alpha}} e^{i(\theta\sigma^m\bar{\theta})\partial_m} &= -\bar{\partial}_{\dot{\alpha}}, \\
e^{i(\theta\sigma^m\bar{\theta})\partial_m} D_\alpha e^{-i(\theta\sigma^m\bar{\theta})\partial_m} &= \partial_\alpha, \\
e^{i(\theta\sigma^m\bar{\theta})\partial_m} \bar{D}_{\dot{\alpha}} e^{-i(\theta\sigma^m\bar{\theta})\partial_m} &= -\bar{\partial}_{\dot{\alpha}} - 2i\theta^\alpha\partial_{\alpha\dot{\alpha}}. \qquad \text{(P35)}
\end{aligned}$$

Problem 2.9. Let $\Phi(x, \theta, \bar{\theta})$ be a chiral superfield with the component decomposition (P34). Show that under the supersymmetry translations

$$\delta\Phi = i(\epsilon^\alpha Q_\alpha + \bar{\epsilon}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}})\Phi \qquad \text{(P36)}$$

the component fields $A(x)$, $\psi_\alpha(x)$, $F(x)$ transform by the rules

$$\begin{aligned}
\delta A(x) &= -\epsilon^\alpha\psi_\alpha(x), \\
\delta\psi_\alpha(x) &= -2\epsilon_\alpha F(x) - 2i\bar{\epsilon}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}}A(x), \\
\delta F(x) &= -i\bar{\epsilon}_{\dot{\alpha}}(\tilde{\sigma}^m)^{\alpha\dot{\alpha}}\partial_m\psi_\alpha(x). \qquad \text{(P37)}
\end{aligned}$$

Problem 2.10. Show that the expression

$$\delta^2(\theta - \theta') \equiv (\theta - \theta')^2 = (\theta - \theta')^\alpha(\theta - \theta')_\alpha \qquad \text{(P38)}$$

satisfy the definition of delta-function with respect to the anticommuting variables

$$\int d^2\theta' \delta^2(\theta - \theta')F(\theta') = F(\theta), \qquad \text{(P39)}$$

Where $F(\theta)$ is an arbitrary superfunction, $d^2\theta = \frac{1}{4}\varepsilon^{\alpha\beta}d\theta_\alpha d\theta_\beta$, $\int d\theta_\alpha\theta^\beta = \delta_\alpha^\beta$.

6.3 Superfield models

Problem 3.1. Prove the identity

$$\int d^4x d^2\bar{\theta}\bar{\mathcal{L}}_c = \int d^4x(-\frac{1}{4}\bar{D}^2)\bar{\mathcal{L}}_c \tag{P40}$$

for arbitrary antichiral superfield $\bar{\mathcal{L}}_c$.

Problem 3.2. Let $V(x, \theta, \bar{\theta})$ be a real scalar superfield with the following classical action

$$S[V] = \frac{1}{8}\int d^8z V D^\alpha \bar{D}^2 D_\alpha V + m^2 \int d^8z V^2. \tag{P41}$$

Prove the following statements.

a. The action (P41) can be represented in the form

$$S[V] = \frac{1}{2}\int d^6z W^\alpha W_\alpha + m^2 \int d^8z V^2, \tag{P42}$$

where $W_\alpha = -\frac{1}{4}\bar{D}^2 D_\alpha V$.

b. The action (P41) leads to the following equation of motion for the field V

$$\frac{1}{8}D^\alpha \bar{D}^2 D_\alpha V + m^2 V = 0. \tag{P43}$$

c. The equation (P43) has the following differential consequences

$$D^2 V = 0, \qquad \bar{D}^2 V = 0. \tag{P44}$$

Hint: use the identities $D^2 \bar{D}_{\dot{\alpha}} D^2 = 0$, $\bar{D}^2 D_\alpha \bar{D}^2 = 0$.

d. The equation of motion (P43) leads to the mass-shell condition for the superfield V

$$(\Box - m^2)V = 0. \tag{P45}$$

Hint: apply the identity $D^2\bar{D}^2 + \bar{D}^2D^2 - 2D^\alpha\bar{D}^2D_\alpha = 16\Box$.

Problem 3.3. Consider the following operators

$$\begin{aligned} \mathcal{D}_\alpha &= e^{-2V}D_\alpha e^{2V}, \qquad \bar{\mathcal{D}}_{\dot\alpha} = \bar{D}_{\dot\alpha}, \\ \mathcal{D}_{\alpha\dot\alpha} &= \frac{i}{2}\{\mathcal{D}_\alpha, \mathcal{D}_{\dot\alpha}\}, \end{aligned} \tag{P46}$$

where V is a real scalar superfield.

a. Check that under the gauge transformations

$$e^{2V'} = e^{i\bar\Lambda}e^{2V}e^{-i\Lambda}, \qquad \bar{D}_{\dot\alpha}\Lambda = 0, \tag{P47}$$

the operators (P46) transform by the rules

$$\mathcal{D}'_A = e^{i\Lambda}\mathcal{D}_A e^{-i\Lambda}, \tag{P48}$$

where $A = \alpha,\ \dot\alpha,\ \alpha\dot\alpha$.

b. Let us introduce the superfields Γ_A by the following relations

$$\mathcal{D}_A = D_A + i\Gamma_A, \tag{P49}$$

where $D_A = (D_\alpha, \bar{D}_{\dot\alpha}, \partial_{\alpha\dot\alpha})$. Show that

$$i\Gamma_\alpha = e^{-2V}(D_\alpha e^{2V}), \qquad \Gamma_{\alpha\dot\alpha} = \frac{i}{2}\bar{D}_{\dot\alpha}\Gamma_\alpha. \tag{P50}$$

c. Prove the relations

$$\{\mathcal{D}_\alpha, \mathcal{D}_\beta\} = 0, \qquad [\bar{\mathcal{D}}_{\dot\alpha}, \mathcal{D}_{\beta\dot\beta}] = 2i\varepsilon_{\dot\alpha\dot\beta} W_\beta, \tag{P51}$$

where $W_\beta = -\frac{1}{8}\bar{D}^2 e^{-2V}(D_\beta e^{2V})$.

Problem 3.4. Consider the model of chiral and antichiral superfields with the action

$$S[\Phi, \bar\Phi, V] = \int d^8 z \bar\Phi e^{2V} \Phi, \tag{P52}$$

where V is an external abelian superfield.

a. Show that the equation of motion for the field Φ in the model (P52) is

$$-\frac{1}{4}D^2 e^{2V}\Phi = 0. \tag{P53}$$

b. Using the relations (P51) show that the equation of motion (P53) can be rewritten as

$$[\mathcal{D}^m \mathcal{D}_m - W^\alpha \mathcal{D}_\alpha - \frac{1}{2}(\mathcal{D}^\alpha W_\alpha)]\Phi = 0, \tag{P54}$$

where $\mathcal{D}_m = -\frac{1}{2}(\sigma_m)_{\alpha\dot\alpha}\mathcal{D}^{\alpha\dot\alpha}$.

Problem 3.5. Let χ_α, $\bar\chi_{\dot\alpha}$ be some spinorial (anticommuting) superfields satisfying the (anti)chirality conditions $\bar{D}_{\dot\alpha}\chi_\alpha = 0$, $D_\alpha \bar\chi_{\dot\alpha} = 0$. Let the action is given by

$$S[\chi, \bar\chi] = -\int d^8 z G^2 - m^2 \int d^6 z \chi^2 - m^2 \int d^6 \bar{z} \bar\chi^2, \tag{P55}$$

where $G = \frac{1}{2}(D^\alpha \chi_\alpha + \bar{D}_{\dot\alpha}\bar\chi^{\dot\alpha}) = \bar{G}$.

a. Prove the relations

$$D^2 G = 0, \qquad \bar{D}^2 G = 0. \tag{P56}$$

b. Show that the action (P55) leads to the following equations of motion

$$\frac{1}{8}\bar{D}^2 D_\alpha G + m^2 \chi_\alpha = 0, \qquad \frac{1}{8} D^2 \bar{D}_{\dot\alpha} G + m^2 \bar{\chi}_{\dot\alpha} = 0. \tag{P57}$$

c. Show that on the equations of motion (P57) the superfields χ_α, $\bar{\chi}_{\dot\alpha}$ satisfy the constraint

$$D^\alpha \chi_\alpha = \bar{D}_{\dot\alpha} \bar{\chi}^{\dot\alpha}. \tag{P58}$$

Hint: apply the identity $D^\alpha \bar{D}^2 D_\alpha = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha}$.

d. Show that the equations of motion (P57) are equivalent to the equations

$$(\Box - m^2)\chi_\alpha = 0, \qquad \bar{\chi}_{\dot\alpha} = -\frac{i}{4m^2}\partial_{\alpha\dot\alpha} D^2 \chi^\alpha. \tag{P59}$$

Hint: use the following identity for the chiral superfield χ_α

$$\bar{D}^2 D^2 \chi_\alpha = 16 \Box \chi_\alpha. \tag{P60}$$

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